1. Structures and theories

Languages

A **language** (**signature**) $\mathcal{L}$ is a set of

1. function symbols f, each with an integer arity $n_f \geq 1$,
2. relation symbols R, each with an integer arity $n_R \geq 1$, and
3. constant symbols (function symbols of arity 0).

Example

1. $\mathcal{L}_{\mathrm{Graph}} = \{E\}$
2. $\mathcal{L}_{\mathrm{Group}} = \{\cdot, ^{-1}, 1\}$
3. $\mathcal{L}_{\mathrm{Ring}} = \{+, -, 0, \cdot, 1\}$
4. $\mathcal{L}_{\mathrm{LO}} = \{<\}$
5. $\mathcal{L}_{\mathrm{ORing}} = \mathcal{L}_{\mathrm{Ring}} \cup \mathcal{L}_{\mathrm{LO}}$
6. $\mathcal{L}_{\mathrm{Set}} = \{\in\}$
7. $\mathcal{L}_{\mathrm{PSet}} = \emptyset$

Here $\cdot, +$ are binary function symbols, $^{-1}, -$ are unary function symbols, $E, <, \in$ are binary relation symbols, $0, 1$ are constants.

Structures

For a language $\mathcal{L}$, an $\mathcal{L}$-**structure** $\mathcal{M}$ is given by

1. a nonempty set M, the **universe** (**domain**) of $\mathcal{M}$,
2. a function $f^{\mathcal{M}}\colon M^{n_f} \to M$ for each function symbol $f \in \mathcal{L}$,
3. a set $R^{\mathcal{M}} \subseteq M^{n_R}$ for each relation symbol $R \in \mathcal{L}$,
4. an element $c^{\mathcal{M}} \in M$ for each constant symbol $c \in \mathcal{L}$.

$f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}}$ are the **interpretations** of the symbols f, R, c in $\mathcal{M}$.

Conventions

We write $\mathcal{M} = (M, f_1^{\mathcal{M}}, \dots, R_1^{\mathcal{M}}, \dots, c_1^{\mathcal{M}}, \dots)$, omitting superscripts when the structure is clear from context.
$A, B, M, N, \dots$ denote the universes of $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \dots$
Structures without relation symbols are **algebras**.
Structures without function symbols are **relational structures**.

Example: Groups

Choose an appropriate language, like $\mathcal{L}_{\text{Group}} = \{\cdot, ^{-1}, 1\}$ for $\cdot$ binary, $^{-1}$ unary, 1 a constant.

1. $(\mathbb{R}, \cdot, ^{-1}, 1)$ is an $\mathcal{L}_{\text{Group}}$-structure with the usual interpretation of function symbols.
2. $(\mathbb{Z}, +, -, 0)$ is an $\mathcal{L}_{\text{Group}}$-structure where we interpret $\cdot$ as addition, $^{-1}$ as additive inverse, 1 as zero.
3. $(\mathbb{N}, +, 0, 0)$ where we interpret $^{-1}$ as constant zero map is an $\mathcal{L}_{\text{Group}}$-structure, but not a group.

Note: Languages are pure syntax. Symbols are not associative, etc. Their interpretation for a specific structure may or may not be.

Example: Graphs, order

- A **(simple undirected) graph** (V, E) consists of a set of vertices V and a set of edges E (2-element subsets of V), hence can be modelled by a structure with universe V and a single binary relation.
- A **(partial) order**, like $(\mathbb{Z}, <)$, can be modelled by a structure with universe $\mathbb{Z}$ and a single binary relation $\{(a, b) \in \mathbb{Z}^2 \ : \ a < b\}$.

Substructures

An $\mathcal{L}$-structure $\mathbf{A}$ is a **substructure** of an $\mathbf{L}$-structure $\mathbf{B}$ ($\mathbf{B}$ is an **extension** of $\mathbf{A}$) if

1. $A \subseteq B$,
2. for each relation $R \in \mathcal{L}$ and all tuples a over A,
   $a \in R^{\mathbf{A}}$ iff $a \in R^{\mathbf{B}}$,
3. for each function $f \in \mathcal{L}$ and all tuples a over A we have
   $f^{\mathbf{A}}(a) \in A$,
4. for each constant $c \in \mathcal{L}$ we have $c^{\mathbf{A}} = c^{\mathbf{B}}$.

Example

- For our choice $\mathcal{L}_{\mathrm{Group}}$, subgroups of a group
- Subgraphs of a graph

Homomorphisms

For $\mathcal{L}$-structures $\mathbf{A}, \mathbf{B}$, an $\mathcal{L}$-**homomorphism** h from $\mathbf{A}$ to $\mathbf{B}$ is a map $h\colon A \to B$ that **preserves** functions, relations, constants in $\mathcal{L}$, i.e.,

1. for each relation $R \in \mathcal{L}$ and every tuple $(a_1, \ldots, a_{n_R})$ over A, if $(a_1, \ldots, a_{n_R}) \in R^{\mathbf{A}}$, then $(h(a_1), \ldots, h(a_{n_R})) \in R^{\mathbf{B}}$,
2. for each function $f \in \mathcal{L}$ and every tuple $(a_1, \ldots, a_{n_f})$ over A we have $h(f^{\mathbf{A}}(a_1, \ldots, a_{n_f}) = f^{\mathbf{B}}(h(a_1), \ldots, h(a_{n_f})),$
3. for each constant $c \in \mathcal{L}$ we have $h(c^{\mathbf{A}}) = c^{\mathbf{B}}$.

We write $h\colon \mathbf{A} \to \mathbf{B}$ for homomorphisms.
$h\colon \mathbf{A} \to \mathbf{B}$ is an $\mathcal{L}$-**embedding** if h is injective and for each relation $R \in \mathcal{L}$ and every tuple $(a_1, \ldots, a_{n_R})$ over A

$$(a_1, \ldots, a_{n_R}) \in R^{\mathbf{A}} \text{ iff } (h(a_1), \ldots, h(a_{n_R})) \in R^{\mathbf{B}}$$

(h preserves and **reflects** R)

Example

- Injective group homomorphisms
- Bijective graph homomorphisms

Terms

Inductive definition: The set of $\mathcal{L}$-**terms** is the smallest set $\mathcal{T}$ such that

1. $c \in \mathcal{T}$ for each constant symbol $c \in \mathcal{L}$,
2. each variable symbol $x_i \in \mathcal{T}$ for $i = 1, 2, \ldots,$
3. for each function symbol $f \in \mathcal{L}$ and $t_1, \ldots, t_{n_f} \in \mathcal{T}$, also $f(t_1, \ldots, t_{n_f}) \in \mathcal{T}$.

Example

- Polynomials over a ring $\mathcal{R}$
- $\mathcal{L}_{\mathrm{Graph}}$-terms

Induced term functions

For an $\mathcal{L}$-structure $\mathbf{A}$ and variables $x_1, \ldots, x_n$, any term $t(x_1, \ldots, x_n)$ induces a function

$$t^{\mathbf{A}} \colon A^n \to A$$

as follows:

1. if t is a constant $c \in \mathcal{L}$, then $t^{\mathbf{A}}(a_1, \ldots, a_n) = c^{\mathbf{A}}$;
2. if t is a variable x_i, then $t^{\mathbf{A}}(a_1, \ldots, a_n) = a_i$;
3. if $t = f(t_1, \ldots, t_{n_f})$ for a function $f \in \mathcal{L}$, then $t^{\mathbf{A}}(a_1, \ldots, a_n) = f^{\mathbf{A}}(t_1^{\mathbf{A}}(a_1, \ldots, a_n), \ldots, t_{n_f}^{\mathbf{A}}(a_1, \ldots, a_n))$

for all $a_1, \ldots, a_n \in A$.

Example

Terms x^p and x

Formulas

What can we express in $\mathcal{L}$?

Recursive definition: An $\mathcal{L}$-**formula** in variables $x_1, \ldots, x_n$ is one of the following:

1. $t_1 = t_2$ for $\mathcal{L}$-terms t_1, t_2 in $x_1, \ldots, x_n$,
2. $R(t_1, \ldots, t_{n_R})$ for a relation $R \in \mathcal{L}$ and $\mathcal{L}$-terms $t_1, \ldots, t_{n_R}$ in $x_1, \ldots, x_n$,
3. $\neg\phi$ for a formula ϕ in $x_1, \ldots, x_n$,
4. $\phi \wedge \psi$ for formulas ϕ, ψ in $x_1, \ldots, x_n$,
5. $\exists y\ \phi$ for a formula ϕ in distinct variables $x_1, \ldots, x_n, y$.

A variable y is **bound** in an $\mathcal{L}$-formula ψ if some subformula of ψ is of the form $\exists y\ \phi$.
For a formula ψ in $x_1, \ldots, x_n$, no variable $x_1, \ldots, x_n$ is bound; they are **free** in ψ.
A **sentence** is a formula without free variables.

Shorthand notation

We write

1. $\phi \vee \psi$ for $\neg(\neg\phi \wedge \neg\psi)$
2. $\forall y\ \phi$ for $\neg(\exists y\ \neg\varphi)$
3. $\phi \rightarrow \psi$ for $\neg\phi \vee \psi$
4. $\phi \leftrightarrow \psi$ for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$
5. $\bigwedge_{i=1}^{k} \phi_i$ for $(\ldots((\phi_1 \wedge \phi_2) \wedge \phi_3) \cdots \wedge \phi_k)$
6. $\bigvee_{i=1}^{k} \phi_i$ for $(\ldots((\phi_1 \vee \phi_2) \vee \phi_3) \cdots \vee \phi_k)$
7. $\bot$ for $\exists x\ x \neq x$
8. $\top$ for $\neg\bot$

Example

Some $\mathcal{L}_{\mathrm{ORing}}$-formulas

- $x_1 = 0 \vee x_1 > 0$
- $\exists x_1\ x_1 \cdot x_1 = -1$
- $\forall x_1 (x_1 = 0 \vee \exists x_2\ x_1 x_2 = 1)$