Math 2001 - Assignment 9

Due March 25, 2026

(1) Give a quantifier-free formula which is equivalent to $\exists x\ ax^2 + bx + c = 0$ over
  (a) $(\mathbb{C}, +, \cdot)$,
  (b) $(\mathbb{R}, +, \cdot)$.
  Hint: Recall that on these structures $-, 0, 1$ are definable. Use them if needed.

(2) Let $\mathcal{A} \leq \mathcal{B}, \mathcal{C}$ be structures and let $h\colon \mathcal{A} \to \mathcal{C}$ be an embedding. Show that for every quantifier free formula $\phi(\bar{x})$ and every $\bar{a}$ over A,
$$\mathcal{B} \models \phi(\bar{a}) \text{ iff } \mathcal{C} \models \phi(h(\bar{a})).$$

(3) Let $\mathcal{A}, \mathcal{B}$ be $\mathcal{L}$-structures. Show that if $\bar{a} \in A^n$ and $\bar{b} \in B^n$ satisfy the same quantifier-free formulas, then the substructure $\langle \bar{a} \rangle$ of $\mathcal{A}$ generated by the entries of $\bar{a}$ is isomorphic to the substructure $\langle \bar{b} \rangle$ of $\mathcal{B}$ generated by the entries of $\bar{b}$.

(4) Let $s\colon \mathbb{N} \to \mathbb{N},\ x \mapsto x + 1$, be the successor function. Show that
  (a) the theory of $(\mathbb{N}, s)$ does not have quantifier elimination,
  (b) but the theory of $(\mathbb{N}, s, 0)$ has quantifier elimination.