

Math 2001 - Assignment 7

Due March 4, 2026

- (1) Which of the following are types of the respective structures? For those which are, determine the elements in that structure that realize them.
- (a) $\{\exists y xy = 1\}$ for $(\mathbb{N}, \cdot, 1)$,
 - (b) $\{x > 0, x > 1, x > 2, \dots\}$ for $(\mathbb{Z}, <)_\mathbb{N}$,
 - (c) $\{x > 0, x < 1\}$ for $(\mathbb{Z}, <)_\mathbb{N}$,
 - (d) $\{x > 0, x < 1, x < \frac{1}{2}, x < \frac{1}{3}, \dots\}$ for $(\mathbb{Q}, <)_\mathbb{Q}$.

For the following fix an \mathcal{L} -theory T .

- Let $S_n(T)$ be the set of all complete n -types of T .
- For an \mathcal{L} -formula $\phi(\bar{x})$, let $[\phi]$ denote the set of complete n -types of T that contain ϕ . Recall that these are the basic open sets of the Stone topology on $S_n(T)$.

- (2) Show that $[\phi] = [\psi]$ iff ϕ and ψ are *equivalent modulo T* , i.e.,

$$T \models \forall \bar{x} (\phi(\bar{x}) \leftrightarrow \psi(\bar{x})).$$

- (3) Complete the proof from class that the sets $[\phi]$ are closed under the Boolean operations:
- (a) $[\phi] \cup [\psi] = [\phi \vee \psi]$
 - (b) $[\phi] \cap [\psi] = [\phi \wedge \psi]$
 - (c) $S_n(T) \setminus [\phi] = [\neg\phi]$
 - (d) $S_n(T) = [\top]$
 - (e) $\emptyset = [\perp]$

- (4) (For students who have taken some basic topology)

Show that every clopen subset of $S_n(T)$ is of the form $[\phi]$ for some formula ϕ .

Hint: For a clopen c , both c and its complement $S_n(T) \setminus c$ are open, hence unions of basic opens sets. Use this to get an open cover of $S_n(T)$, apply compactness and (3).