Math 2001 - Assignment 5

Due February 18, 2026

(1) Let $f\colon \mathcal{A} \to \mathcal{B}$ and $g\colon \mathcal{B} \to \mathcal{C}$ be embeddings. Show that if g and gf are elementary embeddings, then also f is an elementary embedding.

(2) Recall: the language of vector spaces over a field F consists of $+$ and for every $a \in F$ a unary symbol s_a that interprets as scaling a vector x by a.
   (a) For an F-vector space V with basis B, give $|V|$ in terms of $|F|$ and $|B|$.
   (b) For which infinite cardinalities κ is the theory T of F-vector spaces κ-categorical?
   (c) Why is T not complete? Describe the completions of T (i.e. the complete theories that contain T) by a model each.

   Careful about the case that F is finite!

(3) Which of the following theories are $\aleph_0$-categorical?
   (a) ACF_p for $p = 0$ or p prime,
   (b) $\mathrm{Th}(\mathbb{Z}, <)$

(4) [1, Exercise 2.3.3] (For students who have taken Graduate Algebra 2)

   A $\forall\exists$-sentence is of the form $\forall \bar{x} \exists \bar{y}\ \psi(\bar{x}, \bar{y})$ for an atomic formula ψ.

   Show that an $\forall\exists$-sentence ϕ in the language of rings, which is true for all finite fields, is true in all algebraically closed fields.

   Hint: Show that ϕ holds in the algebraic closure of every field F_p of prime order p.

References

[1] Tent, Ziegler. A Course in Model Theory. Lecture Notes in Logic, Cambridge, 2012.