Math 2001 - Assignment 4

Due February 11, 2026

(1) [1, Exercise 2.2.3] Show that the class of connected graphs is not an elementary class.

A graph (V, E) is *connected* if for any vertices $x, y \in V$ there exists a sequence of vertices $x = x_0, x_1, \ldots, x_n = y$ such that $(x_{i-1}, x_i) \in E$ for all $1 \leq i \leq n$.

(2) Show that the interpretation of constants and relations in ultraproducts is well-defined.

(3) Complete the proof of Łoś's Theorem from class.

(4) [1, Exercise 2.1.2] For a class $\mathcal{K}$ of $\mathcal{L}$-structures, let

$$\mathrm{Th}(\mathcal{K}) := \{\phi \;:\; \mathcal{A} \models \phi \text{ for all } \mathcal{A} \in \mathcal{K}\}$$

be the theory of $\mathcal{K}$.

Show that $\mathcal{M}$ is a model for $\mathrm{Th}(\mathcal{K})$ iff $\mathcal{M}$ is elementary equivalent to an ultraproduct of structures in $\mathcal{K}$.

Hint for $\Rightarrow$: Assume $\mathcal{K} = \{\mathcal{A}_i \;:\; i \in I\}$ is a set. For $\mathcal{M}$ a model of $\mathrm{Th}(\mathcal{K})$, choose an ultrafilter $\mathcal{F}$ on I containing $\mathcal{F}_\phi := \{i \in I \;:\; \mathcal{A}_i \models \phi\}$ for every $\phi \in \mathrm{Th}(\mathcal{M})$.

(5) (a) Is $(\mathbb{Z}, <)$ an elementary substructure of $(\mathbb{Q}, <)$?

(b) Is $(\mathbb{Q}, +, \cdot, 0, 1)$ an elementary substructure of $(\mathbb{R}, +, \cdot, 0, 1)$?

References

[1] Tent, Ziegler. A Course in Model Theory. Lecture Notes in Logic, Cambridge, 2012.