Math 2001 - Assignment 3

Due February 4, 2026

(1) The characteristic of a field $\mathcal{F}$ is the smallest positive integer n such that
$$\underbrace{1 + \cdots + 1}_{n\text{times}} = 0$$
if such an n exists; else the characteristic of $\mathcal{F}$ is 0.

Let T be a theory that contains the theory of fields.

(a) Show that if T has models of arbitrarily large positive characteristic, then T has a model of characteristic 0.

(b) Show that there exists no finite theory whose models are exactly the fields of characteristic 0.

(2) (A non-standard model $\mathcal{M}$ for arithmetic) Let $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1, <)$ with the standard interpretation of symbols. Construct a countable structure $\mathcal{M}$ over $\mathcal{L} = \{+, \cdot, 0, 1, <)$ such that $\mathcal{M} \equiv \mathbb{N}$ but $\mathcal{M} \not\cong \mathbb{N}$ as follows:

(a) Let $\mathcal{L}'$ be the expansion of $\mathcal{L}$ by a new constant symbol c, and let
$$S := \{1 < c, 1 + 1 < c, 1 + 1 + 1 < c, \dots\}.$$
Show that $\mathrm{Th}(\mathbb{N}) \cup S$ has a countable model $\mathcal{M}'$.

(b) Show that the $\mathcal{L}$-reduct $\mathcal{M}$ of $\mathcal{M}'$ is elementary equivalent to $\mathbb{N}$.

(c) Show that $\mathcal{M}$ is not isomorphic to $\mathbb{N}$.

(3) For constant symbols $c_1, c_2, \dots$, let $\mathcal{A}$ be the expansion of $(\mathbb{Q}, <)$ with $c_n^{\mathcal{A}} = \frac{1}{n}$ for $n \geq 1$.

Show that there exists a model $\mathcal{B}$ of $\mathrm{Th}(\mathcal{A})$ with an element $\varepsilon \in B$ such that $\mathcal{B} \models 0 < \varepsilon$ and $\mathcal{B} \models \varepsilon < c_n$ for all integers $n \geq 1$.

(4) [1, Ex. 2.2.2] A class $\mathcal{C}$ of $\mathcal{L}$-structures is *finitely axiomatisable* if $\mathcal{C}$ is the class of models of a finite theory.

Show that $\mathcal{C}$ is finitely axiomatisable iff both $\mathcal{C}$ and its complement $\mathcal{C}'$ in the class of all $\mathcal{L}$-structures forms an elementary class.

Hint: For $\Leftarrow$ use the Compactness Theorem.

References

[1] Tent, Ziegler. A Course in Model Theory. Lecture Notes in Logic, Cambridge, 2012.