Math 4140 Spring 2024 - Review

1. **Group algebras.**
   (1) group algebra $\mathbb{C}G$ (6), $\mathbb{C}G$-modules, $\mathbb{C}G$-homomorphisms (4), direct sums of modules (7)
   (2) Maschke's Theorem (8), Schur's Lemma (9), simple $\mathbb{C}G$-modules (10)
   (3) $\mathbb{C}G$ as direct product of matrix rings,
   decomposition of the regular $\mathbb{C}G$-module,
   class number of $G = \dim Z(\mathbb{C}G)$ (12)
   (4) tensor product (19)

2. **Representations.**
   (1) representations of groups and algebras (3,4)
   (2) correspondence between representations, characters and $\mathbb{C}G$-modules (4):
   equivalent representations - equal characters - isomorphic modules,
   irreducible representations - irreducible characters - simple modules
   permutation rep - number of fixed points - permutation module (4)

3. **Characters.**
   (1) regular character and its decomposition (13)
   (2) class functions, inner product, $\mathrm{Irr}G$ as orthonormal basis (14)
   (3) conjugacy classes of S_n and A_n (12)
   (4) row and column orthogonality, character tables (16)
   (5) new characters from old:
   lifting from G/N (17)
   products of characters on G, characters of $G \times H$ (19)
   restriction to subgroups (20)
   inducing characters from subgroups (21)
   (6) Frobenius reciprocity (21)
   (7) ring of algebraic integers, character values are algebraic integers,
   $\chi(1)$ divides $|G|$ for all $\chi \in \mathrm{Irr}G$ (22)
   (8) Burnside's $p^a q^b$ Theorem (31)

4. **Character table and group structure.**
   (1) sizes of centralizers/conjugacy classes are determined by $\mathrm{Irr}G$
   (2) normal subgroups as intersection of character kernels and their sizes
   (3) derived subgroup G' as intersection of the kernels of linear characters,
   $|G : G'| =$ number of linear characters of G
   (4) center $Z(G) = \bigcap_{\chi \in \mathrm{Irr}(G)} \{g \in G \ : \ |\chi(g)| = \chi(1)\}$
   (5) whether G is solvable (or nilpotent) is determined by its character table,
   but the isomorphism type of G is not

References

[1] G. James and M. Liebeck. *Representations and characters of groups.* Cambridge University Press, second edition, 2001.