Math 4140 - Assignment 12

Due April 22, 2024

(1) Let $H \leq G$, let φ be a class function of H and ψ a class function of G. Show
$$(\varphi\psi|_H)^G = \varphi^G\psi.$$
Hint: Consider the inner product of both sides with $\chi \in \mathrm{Irr}G$ and use Frobenius Reciprocity.

(2) (Bonus) Let $H \leq G$, let θ be a character of H. Show
$$\ker\theta^G = \bigcap_{x\in G} (\ker\theta)^x.$$
Hint: Recall $|\theta^\circ(x^{-1}gx)| \leq \theta(1)$.

(3) [1, Exercise 21.2]

(4) [1, Exercise 21.5]

(5) [1, Exercise 21.6]

(6) For $n \geq 2$, let σ be the sign character of S_n and let $\psi \in \mathrm{Irr}A_n$. Show that either
   (a) $\psi^{S_n} \in \mathrm{Irr}S_n$ with $\psi^{S_n}(1) = 2\psi(1)$ and $\sigma\psi^{S_n} = \psi^{S_n}$ or
   (b) ψ^{S_n} is the sum of two distinct irreducible characters $\chi + \chi'$ of S_n with $\psi(1) = \chi(1) = \chi'(1)$ and $\chi' = \sigma\chi$.

   Hint: Use the results from Chapter 20 and Frobenius Reciprocity.

(7) The degrees of the irreducible characters of S_5 are $1, 1, 4, 4, 5, 5, 6$ and the degrees of the irreducible characters of A_5 are $1, 3, 3, 4, 5$. Use (6) to express ψ^{S_5} as sum of irreducible characters for each $\psi \in \mathrm{Irr}A_5$.

   Hint: No need to look up the character tables.

References

[1] G. James and M. Liebeck. Representations and characters of groups. Cambridge University Press, second edition, 2001.