Math 4140 - Assignment 10

Due April 5, 2024

(1) Show that the entries for the character table of the symmetric group S_n are all real.

Note: In fact they can be shown to be integers.

Solution. In S_n two elements are conjugate iff they have the same cycle structure. Hence any permutation $g \in S_n$ is conjugate to its inverse. Thus

$$\chi(g) = \chi(g^{-1}) = \overline{\chi(g)}$$

is real. □

(2) Complete the character table of S_4 from class, that is,
  (a) lift the irreducible characters from $S_4/V_4 \cong S_3$ to S_4,
  (b) show that the permutation character-1 is irreducible,
  (c) compute the remaining irreducible character using column orthogonality.

Solution. See the table on p. 180 of [1].

(3) Let χ be a character of G such that $\chi(g)$ is a non-negative real for all $g \in G$.
  (a) Show that if χ is irreducible, then χ is trivial.

  Hint: Use row orthogonality.
  (b) Give an example of a non-trivial χ with this property.

Solution. (a) Let χ_1 be the trivial character. Then

$$\langle \chi_1, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) > 0.$$

So χ_1 is a constituent of χ. If χ is irreducible, this means that $\chi = \chi_1$

(b) regular character. □

(4) Show

$$Z(G) = \bigcap_{\chi \in \mathrm{Irr} G} \{g \in G \ : \ |\chi(g)| = \chi(1)\}.$$

Hint: Use Theorem 13.11.

Solution. Let $\mathrm{Irr} G = \{\chi_1, \dots, \chi_k\}$ with corresponding irreducible representations $\varphi_1, \dots, \varphi_k$. By Thm 13.11,

$$|\chi_i(g)| = \chi_i(1) \text{ iff } \varphi_i(g) \text{ is a scalar multiple of the identity matrix .}$$

The latter is equivalent to $\varphi_i(g) \in Z(\varphi_i(G))$ by Schur's Lemma.

Note that $\varphi_i(g) \in Z(\varphi_i(G))$ for all $i \leq k$ iff $\rho(g)$ is central in $\rho(G)$ for the regular representation ρ of G. Since ρ is faithful, this is equivalent to $g \in Z(G)$. □

(5) [1, Exercise 17.2]

Solution. in [1]

(6) [1, Exercise 17.5]. Also determine the center and the derived subgroup of D_8 from its character table.

Solution. Normal subgroups are listed in [1]. Further the derived subgroup is the intersection of the kernels of the linear characters $\chi_1, \ldots, \chi_4$, which yields $D_8' = \{1, a^2\}$.

By (4) we see that $Z(G) = \{1, a^2\}$ as well. □

(7) Consider the character table of a group G. Here $\omega = e^{2\pi i/3}$:

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	ω	ω^2	ω^2	ω
χ_3	1	1	1	ω^2	ω	ω	ω^2
χ_4	3	3	-1	0	0	0	0
χ_5	2	-2	0	-1	-1	1	1
χ_6	2	-2	0	$-\omega$	$-\omega^2$	ω^2	ω
χ_7	2	-2	0	$-\omega^2$	$-\omega$	ω	ω^2

(a) Determine the sizes of all conjugacy classes of G.

Solution. Column orthogonality yields (note $\bar{\omega} = \omega^2$).

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
centralizer size	24	24	4	6	6	6	6
class size	1	1	6	4	4	4	4

(b) Determine all normal subgroups of G (in particular $Z(G)$ and G') and their sizes.

Solution. Take all possible intersections of the kernels of the irreducible characters. There are 2 non-trivial proper subgroups, namely

$\{g_1, g_2\} = \bigcap_{i=1}^4 \chi_i = Z(G)$ of size 2

G' = intersection of kernels of linear characters = union of the classes of g_1, g_2, g_3, which has size $1 + 1 + 6 = 8$

(c) Explain why G is a semidirect product of Sylow subgroups.

Solution. Since G' has size 8 and index 3 in G, it is a normal Sylow 2-subgroup. For H a Sylow 3-subgroup of G, we have $G = G'H$ is a semidirect product.

(d) Explain why G has a quotient isomorphic to A_4.

Solution. $G/Z(G)$ has size 12 and is a semidirect product of a normal subgroup of order 4 and a group of order 3 by (c). Note that $G/Z(G)$ is not abelian because otherwise G would have 12 linear characters. So $G/Z(G)$ must be a non-abelian semidirect product of $\mathbb{Z}_2 \times \mathbb{Z}_2$ by $\mathbb{Z}_3$. The only such group is A_4.

Note that this is the character table of $\mathrm{SL}(2, 3)$.

References

[1] G. James and M. Liebeck. Representations and characters of groups. Cambridge University Press, second edition, 2001.