Math 4140 - Assignment 10

Due April 5, 2024

(1) Show that the entries for the character table of the symmetric group S_n are all real.

Note: In fact they can be shown to be integers.

(2) Complete the character table of S_4 from class, that is,
  (a) lift the irreducible characters from $S_4/V_4 \cong S_3$ to S_4,
  (b) show that the permutation character-1 is irreducible,
  (c) compute the remaining irreducible character using column orthogonality.

(3) Let χ be a character of G such that $\chi(g)$ is a non-negative real for all $g \in G$.
  (a) Show that if χ is irreducible, then χ is trivial.

  Hint: Use row orthogonality.
  (b) Give an example of a non-trivial χ with this property.

(4) Show

$$Z(G) = \bigcap_{\chi \in \mathrm{Irr} G} \{g \in G \ : \ |\chi(g)| = \chi(1)\}.$$

Hint: Use Theorem 13.11.

(5) [1, Exercise 17.2]

(6) [1, Exercise 17.5]. Also determine the center and the derived subgroup of D_8 from its character table.

(7) Consider the character table of a group G. Here $\omega = e^{2\pi i/3}$:

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	ω	ω^2	ω^2	ω
χ_3	1	1	1	ω^2	ω	ω	ω^2
χ_4	3	3	-1	0	0	0	0
χ_5	2	-2	0	-1	-1	1	1
χ_6	2	-2	0	$-\omega$	$-\omega^2$	ω^2	ω
χ_7	2	-2	0	$-\omega^2$	$-\omega$	ω	ω^2

  (a) Determine the sizes of all conjugacy classes of G.
  (b) Determine all normal subgroups of G (in particular $Z(G)$ and G') and their sizes.
  (c) Explain why G is a semidirect product of Sylow subgroups.
  (d) Explain why G has a quotient isomorphic to A_4.

References

[1] G. James and M. Liebeck. Representations and characters of groups. Cambridge University Press, second edition, 2001.