# Math 4140 - Assignment 9

Due March 18, 2024

(1) Show that a character $\chi$ of a group $G$ is a homomorphism iff $\chi$ has degree 1. Such characters are called **linear**.

(2) What if we take the determinant instead of the trace of a representation $\rho$ of a group $G$?
   (a) Show that $\det \rho \colon G \to \mathbb{C},\ x \mapsto \det(x^\rho)$, is a linear character of $G$.
   (b) What is $\det \rho$ of the permutation representation $\rho$ of $S_n$?

(3) Recall that any non-abelian group $G$ of order 8 has a center $Z(G) = \{1, z\}$ of size 2 with quotient $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
   (a) Show that $G$ has a unique non-linear irreducible character $\chi$.
   (b) Show $\chi(1) = 2, \chi(z) = -2$, and $\chi(x) = 0$ for all $x \in G \setminus Z(G)$.

   Hint: Use the regular character of $G$.

   Explain that the dihedral group $D_8$ and the quaternion group $Q_8$ have the same character table.

(4) Let $\chi$ be a faithful irreducible character of a group $G$. Show

$$Z(G) = \{g \in G \ : \ |\chi(g)| = \chi(1)\}.$$

(5) Let $\chi$ be the character of $A_4$ of degree 3 from HW 8.8. Show
   (a) $\chi$ is faithful, i.e., $\ker\chi = 1$,
   (b) $\chi$ is irreducible, i.e., $\langle \chi, \chi \rangle = 1$,
   (c) $\chi$ is the difference of the permutation character of $A_4$ and the trivial character, i.e., $\chi(g) = |\mathrm{fix}(g)| - 1$ for all $g \in A_4$.

   Hint: For computing inner products efficiently, use that characters are constant on conjugacy classes and their sizes.

(6) [1, Exercise 14.1]

(7) [1, Exercise 14.2] Also which representations are irreducible?

(8) By (3) both $D_8$ and $Q_8$ have a unique irreducible representation of degree 2 (up to equivalence) and their characters are equal. Show that their determinants are not.

   Hint: Use the representation $\rho_1$ of $Q_8$ given in [1, Exercise 14.2].

## References

[1] G. James and M. Liebeck. Representations and characters of groups. Cambridge University Press, second edition, 2001.