Math 4140 - Assignment 8

Due March 11, 2024

All representations are over $\mathbb{C}$.

(1) (Bonus) For a group G let G' (the **derived subgroup** of G) be generated by all commutators

$$[g,h] := g^{-1}h^{-1}gh \text{ for } g,h \in G.$$

Show that G' is a the smallest normal subgroup of G with abelian quotient, that is,

(a) G' is normal in G;

(b) for any normal subgroup N of G, we have G/N abelian iff $G' \leq N$.

Why does this imply that the kernel of every degree 1 representation of G contains G'?

(2) Let G be a finite group. Show that $|G : G'|$ is the number of inequivalent representations of degree 1 of G.

Hint: For $N \trianglelefteq G$, a homomorphism $\varphi\colon G/N \to H$ lifts to a homomorphism $\hat{\varphi}\colon G \to H, x \mapsto \varphi(xN)$.

Use this to show that every degree 1 representation of G/G' lifts to a degree 1 representation of G. Then show that G has no other degree 1 representations by problem (1).

(3) Use the description of the conjugacy classes in [1, Example 12.18] to show:

(a) A_4 has no subgroup of order 6. (This would be normal. Why?)

(b) A_5 is simple.

(4) Give representatives for the conjugacy classes of S_6 and of A_6.

(5) Give the irreducible characters of $\mathbb{Z}_4$.

(6) Let $D_{12} = \langle a, b \ : \ a^6 = 1, b^2 = 1, b^{-1}ab = a^{-1}\rangle$, let $\omega := e^{2\pi i/3}$ and let ρ be a representation of D_{12} with

$$a\rho = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad b\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Determine the conjugacy classes of D_{12}, the character of ρ and its kernel.

(7) Let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation by an angle α that fixes an axis through the origin. Explain that there is a basis B of $\mathbb{R}^3$ such

that

$$[\varphi]_B = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Compute its trace.

(8) (Continuation of 7.6c) Recall that the alternating group A_4 has 4 conjugacy classes with representatives $(), (12)(34), (123), (132)$.

Labelling the 4 corners of a regular tetrahedron $1, 2, 3, 4$, one sees that the group of its rotations is isomorphic to A_4. In particular the rotations are:

(a) the identity (corresponding to ()),
(b) rotations by $\pm 2\pi/3$ fixing an axis through one corner and the center of the opposite face (8 elements, corresponding to the 3-cycles),
(c) rotations by π fixing an axis through the midpoints of 2 opposite edges (3 elements, swapping the corners of these edges, like (12)(34)).

Let ρ be the degree 3 representation of A_4 by these rotation matrices. Determine its character χ.

Hint: Don't write down any rotation matrices but use (7).

References

[1] G. James and M. Liebeck. Representations and characters of groups. Cambridge University Press, second edition, 2001.