Math 4140 - Assignment 7

Due March 4, 2024

All representations are over $\mathbb{C}$.

(1) Let U be a simple $\mathbb{C}G$-module. Show that every submodule W of $U \times U$ with $W \cong U$ is of the form

$$0 \times U \text{ or } \{(x, \lambda x) \ : \ x \in U\}$$

for some $\lambda \in \mathbb{C}$.

(2) (Bonus: Uniqueness of direct decomposition) Let $U_1, \dots, U_m$, $W_1, \dots, W_n$ be simple $\mathbb{C}G$-modules such that

$$U_1 \oplus \cdots \oplus U_m \cong W_1 \oplus \cdots \oplus W_n.$$

Show that $m = n$ and there exists $\pi \in S_n$ such that for all $i \leq n$:

$$U_{i\pi} \cong W_i.$$

Hint: Use induction on n.

(3) For $n \geq 3$ odd, show that the dihedral group $D_{2n} = \langle a, b \ : \ a^n = 1, b = 1, b^{-1}ab = a^{-1} \rangle$ has up to equivalence exactly

- 2 irreducible degree 1 representations, the trivial one and σ with $a\sigma = 1, b\sigma = -1$,
- $\frac{n-1}{2}$ irreducible degree 2 representations θ_{ω^k} for $\omega = e^{2\pi i/n}$ and $k \in \{1, \dots, \frac{n-1}{2}\}$ as given in HW 5.3.

Hint: We already know that the representations above are irreducible and pairwise inequivalent. Explain why they are all up to equivalence.

(4) Determine all irreducible representations of D_{2n} for even $n \geq 3$ up to equivalence.

Hint: Recall $D_{2n}/\langle a^2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and the irreducible representations of the latter. Then argue as for the previous problem.

(5) Given a group G of order 12, what are the possible lists of degrees of its irreducible representations (up to equivalence)?

(6) (a) The alternating group A_4 has a normal subgroup V_4 with $A_4/V_4 \cong \mathbb{Z}_3$.
Use this to show that A_4 has exactly 3 irreducible degree 1 representations and 1 irreducible degree 3 representation ρ (up to equivalence).

(b) By HW 5.4 the $\mathbb{C}A_4$-permutation module with basis $b_1, \dots, b_4$ is a direct sum of a 1-dimensional submodule U induced by the trivial representation and a 3-dimensional submodule W. Explain why W is induced by ρ.

(c) The group of rotations of a regular tetrahedron is isomorphic to the alternating group A_4.
This yields a representation θ of A_4 by 3×3 rotation matrices. Explain why θ is equivalent to ρ.

Hint: No computation of actual representations is needed.