Math 4140 - Assignment 6

Due February 26, 2024

All representations are over $\mathbb{C}$.

(1) [1, Exercise 9.3] Show for $r, n_1, \ldots, n_r \in \mathbb{N}$ that $G := \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ has a faithful representation of degree r. Can G have a faithful representation degree less than r?

(2) [1, Exercise 9.7] Which of the following have a faithful irreducible representation?
   (a) $\mathbb{Z}_n$
   (b) $\mathbb{Z}_2 \times \mathbb{Z}_2$
   (c) D_{2n}
   (d) $\mathbb{Z}_2 \times D_8$
   (e) $\mathbb{Z}_3 \times D_8$

   Hint: Note that for the induced $\mathbb{C}G$-module V, multiplication by elements in $Z(G)$ yields a $\mathbb{C}G$-homomorphism.

(3) For $D_6 = \langle a, b \ : \ a^3 = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle$, define

$$u_1 := 1 + a + a^2 + b + ab + a^2 b, \quad u_2 := 1 + a + a^2 - b - ab - a^2 b$$

in $\mathbb{C}D_6$. Show that
   (a) $U_1 := \mathrm{span}(u_1)$ and $U_2 := \mathrm{span}(u_2)$ are submodules of the regular $\mathbb{C}D_6$-module;
   (b) the representations θ_1, θ_2 afforded by U_1, U_2 are not equivalent;
   (c) there are only two homomorphisms $D_6 \to \mathrm{GL}(1, \mathbb{C})$. Hence θ_1, θ_2 are the only degree 1 representations of D_6 and U_1, U_2 are the only dimension 1 $\mathbb{C}D_6$-modules up to isomorphism.

(4) [1, Exercise 10.2] Write the regular $\mathbb{C}\mathbb{Z}_4$-module as a direct sum of irreducible $\mathbb{C}\mathbb{Z}_4$-submodules. Give the bases for the submodules explicitly.

(5) Determine the irreducible representations of D_8 (up to equivalence).

   Hint: Follow the approach of [1, Example 10.8(2)].

(6) For $\mathbb{C}G$-modules V, W, let

$$\mathrm{Hom}_{\mathbb{C}G}(V, W) := \{\varphi \colon V \to W \ : \ \varphi \text{ is a } \mathbb{C}G - \text{homomorphism}\}.$$

   (a) Show that $\mathrm{Hom}_{\mathbb{C}G}(V, W)$ is a vector space over $\mathbb{C}$ with pointwise addition and scalar multiplication of functions.
   (b) Describe the elements in $\mathrm{End}_{\mathbb{C}G}(V) := \mathrm{Hom}_{\mathbb{C}G}(V, V)$ explicitly for simple V.

References

[1] G. James and M. Liebeck. Representations and characters of groups. Cambridge University Press, second edition, 2001.