Math 4140 - Assignment 5

Due February 19, 2024

Problems 1,2,4 could constitute a practice midterm.

(1) Let $G := D_8$ with the usual presentation.

(a) Show that

$$\rho\colon D_8 \to \mathrm{GL}(3,\mathbb{C}) \text{ with } a\rho = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}, b\rho = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}$$

is a representation of G.

(b) Let $V := \mathbb{C}^3$ be the $\mathbb{C}G$-module induced by ρ. Show that

$$\varphi\colon V \to V,\ v \mapsto v \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix},$$

is a $\mathbb{C}G$-homomorphism.

(c) Show that $U := \mathrm{span}((1,0,1))$ is a $\mathbb{C}G$-submodule of V.

(d) Show that the representation that U affords is not trivial.

(2) Recall from class that

$$\rho\colon \mathbb{Z}_3 \to \mathrm{GL}(2,\mathbb{C}) \text{ with } 1\rho = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

defines a representation of $G = (\mathbb{Z}_3,+)$. Let $V = \mathbb{C}^2$ be the $\mathbb{C}G$-module induced by ρ.

Find proper $\mathbb{C}G$-submodules U, W of V such that $V = U \oplus W$ if possible.

(3) For $n \geq 3$ let $D_{2n} = \langle a, b \ :\ a^n = 1, b^2 = 1, b^{-1}ab = a^{-1}\rangle$. Recall that for $\omega \in \mathbb{C}$ such that $\omega^n = 1$,

$$\theta_\omega\colon D_{2n} \to \mathrm{GL}(2,\mathbb{C}) \text{ with } a\theta_\omega = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad b\theta_\omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

defines a representation of D_{2n}.

(a) Show that θ_{ω_1} and θ_{ω_2} are equivalent iff $\omega_2 \in \{\omega_1, \omega_1^{-1}\}$.

(b) Assume that $\omega \neq \pm 1$. Show that $A \in \mathbb{C}^{2\times 2}$ commutes with $a\theta_\omega$ and $b\theta_\omega$ only if $A = \lambda I_2$ for some $\lambda \in \mathbb{C}$.

(c) Show that if $\omega = \pm 1$, then there exists $A \in \mathbb{C}^{2\times 2}$ which commutes with $a\theta_\omega$ and $b\theta_\omega$ but is not a scalar multiple of I_2.

(d) Use the converse of Schur's Lemma and (b), (c) to show that θ_ω is irreducible iff $\omega \neq \pm 1$.

(e) Let $\omega = e^{2\pi i/n}$. Show that θ_ω is faithful. Deduce that $Z((D_{2n})\theta_\omega) \subseteq \{\pm I_2\}$ and that

$$Z(D_{2n}) = \begin{cases} 1 & \text{if } n \text{ odd,} \\ \{1, a^{n/2}\} & \text{if } n \text{ even.} \end{cases}$$

(4) For $G \leq S_n$, let $V = \mathbb{C}^n$ be the $\mathbb{C}G$-permutation module with basis $B = (b_1, \ldots, b_n)$. Show that if $n > 1$, then V is a direct sum of a trivial and a faithful submodule as follows:

(a) Show $U := \operatorname{span}(\sum_{i=1}^n b_i)$ is a trivial $\mathbb{C}G$-submodule. What is $\dim U$?

(b) Show $W := \operatorname{span}(b_1 - b_i \ : \ i \in \{2, \ldots, n\}$ is a faithful $\mathbb{C}G$-submodule. What is $\dim W$?

(c) Show $V = U \oplus W$.

(5) Label the corners of a square by $1, 2, 3, 4$ counterclockwise and let the symmetry group of the square $D_8 = \langle a, b \rangle$ act on them via $a = (1234), b = (14)(23)$.

From this action you get a $\mathbb{C}D_8$-permutation module $V = F^4$ which affords a representation ρ.

(a) Give $a\rho, b\rho$.

(b) Decomposing $V = U \oplus W$ as in (4), let φ be the representation afforded by W. Determine $a\varphi, b\varphi$.

(c) Is φ irreducible?

(d) Try to find a subspace S of F^3 (the $\mathbb{C}G$-module induced by φ) that is an eigenspace for $a\varphi$ and for $b\varphi$. Explain why this would be isomorphic to a $\mathbb{C}G$-submodule of W.

(e) Bonus: If you found S as above, determine a $\mathbb{C}G$-submodule T of F^3 (the $\mathbb{C}G$-module induced by θ) such that $F^3 = S \oplus T$. Determine the representation θ afforded by T.

(6) Write down all the irreducible representations of $\mathbb{Z}_2, \mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_3$ over $\mathbb{C}$.