Math 4140 - Assignment 4

Due February 12, 2024

(1) Cf. Assignment 2(2b). Show that for $m \geq 3$ every faithful representation of $(\mathbb{Z}_m, +)$ over $\mathbb{R}$ of degree 2 is irreducible.
What about over $\mathbb{C}$?

(2) For $n \geq 3$ let $D_{2n} = \langle a, b \ : \ a^n = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle$ and $\omega = e^{i2\pi/n}$. Show that

$$\theta \colon D_{2n} \to \mathrm{GL}(2, \mathbb{C}) \text{ with } a\theta = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad b\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

defines an irreducible representation.

(3) Let $G := \mathbb{Z}_2 \times \mathbb{Z}_2$.
   (a) Determine the matrices for the regular representation ρ of G over $\mathbb{R}$ explicitly.
   (b) Express $\mathbb{R}G$ as a direct sum of 1-dimensional $\mathbb{R}G$-submodules
   (c) Using this decomposition of $\mathbb{R}G$ determine the matrices for a representation θ, which is a sum of degree 1 representations and equivalent to ρ.

(4) Cf. Assignment 2(4). Let $V = U_1 \oplus \cdots \oplus U_r$ be an FG-module that is a direct sum of 1-dimensional FG-submodules $U_1, \ldots, U_r$.
Prove that if V is faithful (i.e., V affords a faithful representation), then G is abelian.

(5) [1, Exercise 8.5] Let

$$G := \left\{ \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \ : \ z \in \mathbb{Z} \right\}$$

act on $\mathbb{C}^2$ by right multiplication, call the corresponding $\mathbb{C}G$-module V.
Show
   (a) G is isomorphic to $\mathbb{Z}$.
   (b) V has only one proper non-trivial $\mathbb{C}G$-submodule.
   (c) V is not completely reducible (This shows that Maschke's Theorem fails for infinite groups).

(6) Let G be a finite group, let F be a field, and let $z := \sum_{g \in G} g \in FG$.
   (a) Show that $zg = gz = z$ for all $g \in G$.

(b) Deduce that

$$\varepsilon\colon FG \to FG,\ a \mapsto az,$$

is an FG-module homomorphism.

(c) Show that for all $\sum_{g\in G} c_g g \in FG$ we have

$$(\sum_{g\in G} c_g g)\varepsilon = (\sum_{g\in G} c_g) z$$

and consequently

$$\mathrm{im}\varepsilon = Fz$$

is an FG-submodule of the regular FG-module.

(d) Show that

$$\ker\varepsilon = \{\sum_{g\in G} c_g g \in FG \ :\ \sum_{g\in G} c_g = 0\}$$

is an FG-submodule of the regular FG-module.

(7) Continued, with the assumption that $\mathrm{char}F$ does not divide $|G|$.

(a) Show that the representation of G corresponding to $\mathrm{im}\varepsilon$ is trivial.

(b) Give a basis for $\ker\varepsilon$.

(c) Show that $FG = \mathrm{im}\varepsilon \oplus \ker\varepsilon$.

What changes if $\mathrm{char}F$ divides $|G|$?

(8) Bonus problem. Continued, with the assumption that $\mathrm{char}F$ does not divide $|G|$.

(a) Show that the **augmentation map**

$$\epsilon\colon FG \to F,\ \sum_{g\in G} c_g g \mapsto \sum_{g\in G} c_g,$$

is a ring homomorphism.

Hence $A = \ker\epsilon$ is an ideal (called the **augmentation ideal**) of the ring FG.

(b) Deduce from (6a) that Fz is also an ideal of FG and that

$$FG = Fz \oplus A \cong F \times A$$

as rings.

(c) Show that for every FG-module V,

$$V = Vz \oplus VA$$

References

[1] G. James and M. Liebeck. Representations and characters of groups. Cambridge University Press, second edition, 2001.