Math 4140 - Assignment 3

Due February 5, 2024

(1) [1, cf. Exercise 6.1] Let $D_8 := \langle a, b \;:\; a^4 = 1, b^2 = 1, b^{-1}ab = a^{-1}\rangle$.
 (a) Compute xy, yx, x^2 for

$$x := a + 2a^2,\ y := 2a - 3b \text{ in } \mathbb{C}D_8.$$

 (b) Let $z = b + a^2b$. Show that $zg = gz$ for all $g \in D_8$. Deduce that $zr = rz$ for all $r \in \mathbb{C}D_8$.

(2) [1, Exercise 4.1] Let $V = \mathbb{R}^3$ be the permutation $\mathbb{R}S_3$-module defined by $e_i g := e_{ig}$ for $g \in G$ and $E = (e_1, e_2, e_3)$ the standard basis of $\mathbb{R}^3$.
 (a) Give $[g]_E$ for all $g \in S_3$.
 (b) For $B = ((1,1,1),(1,-1,0),(1,0,-1))$ give $[g]_B$ for all $g \in S_3$. What do you notice about this representation?

(3) Let A be an F-algebra with zero 0_A, let V be an A-module with zero 0_V. Show
 (a) $0_V a = 0_V$ for all $a \in A$,
 (b) $v 0_A = 0_V$ for all $v \in V$.

(4) Show the lemma from class that representations of G and of FG are essentially the same:
 Let G be a group, F a field, $n \in \mathbb{N}$.
 (a) Every group representation $\rho\colon G \to \mathrm{GL}(n, F)$ extends to an algebra representation

$$\bar{\rho}\colon FG \to F^{n\times n},\ \sum_{x\in G} c_x x \mapsto \sum_{x\in G} c_x x^{\rho}.$$

 (b) Every algebra representation $\sigma\colon FG \to F^{n\times n}$ restricts to a group representation

$$\sigma|_G\colon G \to \mathrm{GL}(n, F).$$

(5) Show the lemma from class that representations of FG and FG-modules are essentially the same:
 Let F a field, A an F-algebra, $n \in \mathbb{N}$.
 (a) Every representation $\sigma\colon A \to F^{n\times n}$ induces an A-module on F^n via

$$va := va^{\sigma} \text{ for } v \in F^n, a \in A.$$

(b) For an A-module V with basis $B = (b_1, \dots, b_n)$, let $[a]_B$ be the matrix of the linear map $V \to V,\ v \mapsto va$, with respect to B. Then

$$\rho\colon A \to F^{n\times n},\ a \mapsto [a]_B,$$

is a representation of A.

References

[1] G. James and M. Liebeck. Representations and characters of groups. Cambridge University Press, second edition, 2001.