Math 4140 - Assignment 2

Due January 31, 2024

(1) Let $m, n \in \mathbb{N}, F$ a field and $A \in \mathrm{GL}(n, F)$. Show that

$$\rho\colon \mathbb{Z}_m \to \mathrm{GL}(n, F), [x] \mapsto A^x,$$

is a representation iff $A^m = I$ (the identity matrix).

(2) Let $m \in \mathbb{N}$. Show

(a) $(\mathbb{Z}_m, +)$ has a faithful representation of degree 1 over $\mathbb{R}$ iff $m \leq 2$.

(b) $(\mathbb{Z}_m, +)$ has a faithful representation of degree 2 over $\mathbb{R}$.

(3) Let $D_{2n} = \langle a, b \ :\ a^n = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle$.

Show that there is a representation

$$\rho\colon D_{2n} \to \mathrm{GL}(1, \mathbb{R}) \text{ with } a\rho = (1), b\rho = (-1).$$

What are $\ker\rho, \mathrm{im}\rho$?

(4) Let ρ be a representation of G of degree 1 over $\mathbb{C}$. Show

(a) $G/\ker\rho$ is abelian.

(b) If G is finite, then $G/\ker\rho$ is cyclic.

(5) The symmetric group S_3 has a presentation

$$S_3 = \langle a, b \ :\ a^2 = 1, b^2 = 1, (ab)^3 = 1 \rangle.$$

Show that there is a representation

$$\rho\colon S_3 \to \mathrm{GL}(2, \mathbb{R}) \text{ with } a\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, b\rho = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}.$$

Is ρ faithful?

(6) Give some faithful representation of D_8 of degree 3 over $\mathbb{C}$.