REVIEW: RELATIONS

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1. Basic properties

Definition. Let A, B be sets. A **relation** R from A to B is a subset of $A \times B$.

For $(a, b) \in R$, we say a and b are related and also write aRb.

If $A = B$, then R is a relation on A.

Example. $R = \{(0,0), (0,1), (1,1)\}$ is a relation on $A = \{0, 1\}$.

Here $0R0, 0R1, 1R$. This relation is also known as $\leq$.

Definition. Let R be a relation on A. Then

(1) R is **reflexive** if $\forall x \in A\colon xRx$
(every element is related to itself)
(2) R is **symmetric** if $\forall x, y \in A\colon xRy \Rightarrow yRx$
(if x is related to y, then also y is related to x)
(3) R is **antisymmetric** if $\forall x, y \in A\colon (xRy \wedge yRx) \Rightarrow x = y$
(x is related to y and y is related to x only if $x = y$)
(4) R is **transitive** if $\forall x, y, z \in A\colon (xRy \wedge yRz) \Rightarrow xRz$

Note that antisymmetric is not the same as 'not symmetric'.

Definition. A relation R on A is

(1) an **equivalence relation** if R is reflexive, symmetric, transitive,
(2) a **partial order** if R is reflexive, antisymmetric, transitive.

Example.

(1) Equivalence relations are used for classifying elements of A. Examples are $=, \equiv_n$ on $\mathbb{Z} (n \in \mathbb{N})$, has the same absolute value on $\mathbb{R}$, has the same cardinality on sets.
(2) Partial orders are used for ordering elements of A. Examples are $=, \leq$ on $\mathbb{R}$, $|$ on $\mathbb{N}$, $\subseteq$ on sets

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2. Equivalences and partitions

Definition. For an equivalence relation R on A and $a \in A$,

$$[a]_R := \{x \in A \ : \ xRa\}$$

is the **equivalence class** of a.

Theorem 1. *Let R be an equivalence on A, let $a, b \in A$. Then*

(1) $[a] = [b]$ *iff* aRb;
(2) $[a] \cap [b] = \emptyset$ *iff* $a \not R b$;
(3) $\bigcup_{a \in A} [a] = A$.

Hence the whole set A is partitioned into disjoint equivalence classes.

Definition. A **partition** of a set A is a set of non-empty subsets $\{A_i \ : \ i \in I\}$ such that

(1) $A_i \cap A_j = \emptyset$ for all distinct $i, j \in I$,
(2) $\bigcup_{i \in I} A_i = A$.

Every equivalence on A gives a partition of A and conversely.

Corollary 2. *Let R be an equivalence on A. Then the set of equivalence classes $\{[a] \ : \ a \in A\}$ is a partition of A.*

Theorem 3. *Let $\{A_i \ : \ i \in I\}$ a partition of A. For $a, b \in A$ define*

$$a \sim b \text{ if } a, b \in A_i \text{ for some } i \in I.$$

Then $\sim$ is an equivalence relation on A with classes $\{A_i \ : \ i \in I\}$.

3. Integers modulo n

One particular important equivalence relation is $\equiv_n$ on $\mathbb{Z}$ for $n \in \mathbb{N}$. The class of $a \in \mathbb{Z}$ is

$$[a] = \{a + zn \ : \ z \in \mathbb{Z}\}.$$

Note $[n] = [0]$. The set of classes

$$\mathbb{Z}_n := \{[0], [1], [2], \ldots, [n-1]\}$$

is called the **integers modulo** n.

Define $+, -, \cdot$ on $\mathbb{Z}_n$ by

$$[a] + [b] := [a + b]$$
$$-[a] := [-a]$$
$$[a] \cdot [b] := [a \cdot b]$$

These operations are well-defined (independent of the choice of representatives for each class) and satisfy the same laws as $+, -, \cdot$ on $\mathbb{Z}$: associativity, commutativity, distributivity, etc.