Math 3140 - Practice Final

(1) (a) Does S_5 contain an element of order 6? What about A_5?
    (b) Give a Sylow 5-subgroup of S_5.

(2) Let G be a group. Show that $\varphi\colon G \to G, x \mapsto x^{-1}$, is a homomorphism if and only if G is abelian.

(3) Prove or provide a counter-example: Let G be a group. If N is an abelian normal subgroup of G and G/N is abelian, then G is abelian.

(4) Let $K := \langle (1\ 2)(3\ 4), (1\ 3)(2\ 4) \rangle$. Determine $|K|$ and the index of K in S_4. What is K isomorphic to?

(5) In how many ways can the vertices of a pentagon be colored in at most 2 colors up to symmetry?

(6) Let $n \in \mathbb{N}$.
    (a) Give a normal subgroup M of D_{4n} that is isomorphic to $\mathbb{Z}_2$. Explain why M is normal.
    (b) Give a normal subgroup N of D_{4n} that is isomorphic to D_{2n}. Explain why N is normal.
    (c) What is $M \cap N$? Show that $D_{4n} \cong D_{2n} \times \mathbb{Z}_2$ if n is odd.

(7) (a) How many abelian groups of order 200 are there up to isomorphism?
    (b) Show that every group G of order 200 has some non-trivial proper normal subgroup N.

(8) Let $U := \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{R}\}$ denote the subring of $\mathbb{R}^{2\times 2}$ containing all upper triangular matrices.
    (a) Show that

$$\varphi\colon U \to U, \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix},$$

    is a unital ring homomorphism.
    (b) Determine $\ker \varphi$.
    (c) Show that $U/\ker \varphi$ is isomorphic to a direct product of rings. Which?