Rubik's cube versus Sudoku

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1. Rubik's cube

Rubik's cube

Invented by Ernő Rubik (1974).
6 colored, revolving faces

Goal: After arbitrary rotations, return the faces to only show one color again.

Idea: Solving by layers

1. Solve top.
2. Solve sides, fixing the top.
3. Solve bottom, fixing the rest.

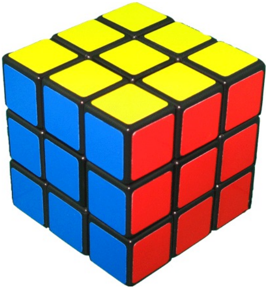
What is the math background?

- A sequence of rotations has to be reversed.
- Solution requires "computations with rotations".
- Abstract Algebra has tools for that (Computational Group Theory).

Rubik's cube as group

6 faces: front, back, up, down, left, right.

up

front  right

f ... rotation of the front face by 90° around the clock, etc.

Rotations f, b, u, d, l, r move the 8 corner and 12 edge pieces. The 6 central pieces remain fixed.

Enumerate the moving pieces [a]:

```
                +--------------+
                |              |
                |  1    2    3 |
                |              |
                |  4   up    5 |
                |              |
                |  6    7    8 |
                |              |
 +--------------+--------------+--------------+--------------+
 |              |              |              |              |
 |  9   10   11 | 17   18   19 | 25   26   27 | 33   34   35 |
 |              |              |              |              |
 | 12  left  13 | 20 front  21 | 28 right  29 | 36  back  37 |
 |              |              |              |              |
 | 14   15   16 | 22   23   24 | 30   31   32 | 38   39   40 |
 |              |              |              |              |
 +--------------+--------------+--------------+--------------+
                |              |
                | 41   42   43 |
                |              |
                | 44  down  45 |
                |              |
                | 46   47   48 |
                |              |
                +--------------+
```

Rotation f yields permutation of pieces.
$f : 17 \to 19 \to 24 \to 22 \to 17, 6 \to 25 \to \ldots$

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[a] Schönert, Rubik's cube in GAP (1993)

What is Computational Group Theory?

Efficient computations with permutations to answer questions like:

1. How many configurations of the cube are there?

$$43\ 252\ 003\ 274\ 489\ 856\ 000$$

2. Can you twist a single corner of the cube?

   No, $(6, 17, 11)$ is not generated by f, b, u, d, l, r.

3. How to write a permutation of the cube as a composition of f, b, u, d, l, r?
   (i.e., how to solve a twisted cube?)

Rubik's cube is in P[b]

1. Computer Algebra can solve arbitrary $n \times n \times n$ cubes efficiently.
2. Time is linear in the number of moving pieces. ($n = 30$ takes ~ 100 times as long as $n = 3$).

[b] solvable in Polynomial time by a deterministic Turing machine (computer)

How would God solve the cube?

- Computed solutions for the $3 \times 3 \times 3$ cube take $60 - 100$ rotations on average.
- What is the smallest number of rotations to solve any given configuration?

20

(Computer proof on Google's hardware[c])

[c] Rokicky, et al. God's number is 20 (2010)

2. Sudoku

Sudoku

9×9 grid subdivided into 3×3 blocks

Each number from 1 to 9 occurs exactly once in

- each row,
- each column,
- each 3×3 block.

Goal: Complete the grid starting from a few given numbers.

	3							
			1	9	5			
		8					6	
8				6				
4			8					1
				2				
	6					2	8	
			4	1	9			5
							7	

Every puzzle has exactly one completion.

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Sudoku history

- The name comes from ‘Suji wa dokushin ni kagiru’ – Japanese for ‘the digits must be single’.
- Grids are special Latin squares (Euler, 18th century).
- Similar games in France ($\sim$ 1900).

eut la tête prise et écrasée par les tampons.
La mort a été instantanée.

Léopold Lepers.

DIVERTISSEMENTS QUOTIDIENS

N° 3879 — CARRÉ MAGIQUE DIABOLIQUE
Par M. B. Meyniel

Compléter le carré ci-dessous en employant les neuf premiers nombres chacun neuf fois de manière que les horizontales, les verticales et les deux grandes diagonales donnent toujours à l'addition le même total.

7	8	9	1	2	3	4	5	6
3				4				8
5				9				1
8				3				4
1	2	3	4	5	6	7	8	9
6				7				2
9				1				5
2				6				7
4	5	6	7	8	9	1	2	3

Ce carré devra être diabolique, c'est-à-dire que le carré restera magique si l'on place une ligne horizontale ou une colonne verticale à la suite de toutes les autres.

N° 3865 — MATHÉMATIQUES
Par M. Adolphe R.
Solution

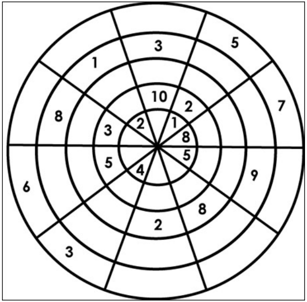
Le marchand a vendu 27,075 vases ; le 95e jour, le dernier, il a vendu 567 vases.

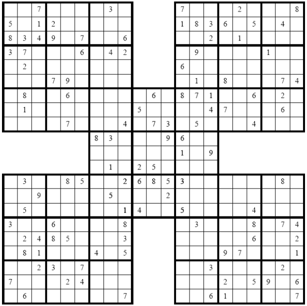
Solutions justes

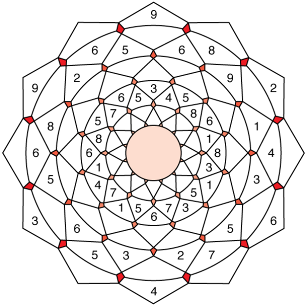
MM. Améthyste ; un chercheur ; Paul et Jules Duplant ; Albert Labarre ; L. Grobet ; C. Gerbaulet.

- Modern version by Howard Garns (Dell Puzzle Magazine, 1979).
- Popular as Sudoku in Japan since 1980s.
- International since 2005.

Fun and games in all variations







What is the math background?

- Not about numbers – they can be replaced by 9 arbitrary distinct symbols.
- Solution does not require calculations but logic.

Combinatorics

1. How many 9×9 Sudokus (full grids) are there?

   6 670 903 752 021 072 936 960[d]

2. How many are really distinct (up to permutation of numbers, rows, columns, rotations, reflections)?

   5 472 730 538[e]

3. How many distinct puzzles (partial grids) are there?

   unknown

[d] Felgenhauer & Jarvis. Enumerating possible Sudoku grids. 2005.

[e] Jarvis & Russell. 2005.

4. How many numbers need to be given to guarantee a unique completion?

17 or more[f]

					2	7	5	
	1	8		9				
4	9							
	3							8
			7			2		
				3				9
7								
5							8	

[f] McGuire, Tugeman, Civario. There is no 16-clue Sudoku. 2014

Backtracking

A simple algorithm in 2 steps:

- Forward: Insert the smallest possible number in the next free spot.
- Back: Replace the last choice by the next largest possible number.

Strategy:

1. Forward as long as possible.
2. If you cannot go forward, the last choice was wrong. Go back and then forward again.

Backtracking for a Sudokid

	1	2	
1			3

Human approach

1. Naked singles: Fix a place. Which number can be inserted?

	1	2	*
1			3

2. Hidden singles: Fix a number. In which spot of a row, column, block can it be inserted?

	1	2	
		2	2
1		2	3

3. Pairs, triples, X-wing, Swordfish, . . .
   backtracking

Hardness of a puzzle is measured by the complexity of the strategies required for solving.
Backtracking is necessary only for the hardest Sudokus.

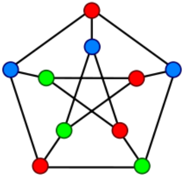
Solutions by computer

- 9×9 Sudokus are easily solved by backtracking.
- For $n \times n$ Sudokus in general there is no efficient algorithm know.
- Time for backtracking is exponential in the number of places to fill.
- n^2 places filled with numbers 1 to n:
  n^{n^2} options

Related problems

$n \times n$ Sudokus are NP-complete problems[g].
Hence they are equivalent to

- Coloring graphs



[g] in NP: solvable by a Non-deterministic Turing machine (computer that can guess) in Polynomial time
NP-complete: the hardest problems in NP

- Satisfiability of Boolean formulas (SAT)

$$(x_1 \wedge x_2) \vee (x_2 \wedge \neg x_3)$$

- Planning schedules
- Travelling Salesman

Real life applications for

- data base queries
- hardware/software verification

NP-complete problems

For all these problems:

1. Given a solution, its correctness can be checked efficiently (in NP).
2. No efficient general algorithm for finding a solutions is known (exponential time).
3. An efficient algorithm for **one problem** (e.g. $n^2 \times n^2$ Sudoku) would also solve **all others** efficiently (NP-hard).

P versus NP (Cook, 1971):

The most famous open question in computer science:

> **Is a problem for which solutions are easily verifiable also easily solvable?**

1 million Dollar for the answer (Millenium Prize of the Clay Mathematical Institute).

P vs. NP (Cook, 1971):

The most famous open question in computer science (alternative formulation):

Are Sudokus harder than Rubik's cube?

1 million Dollar for the answer (Millenium Prize of the Clay Mathematical Institute).

Research at CU Boulder, Math Dept

At the foundations group (Bulin, Kearnes, Mayr, Steindl, Szendrei,. . .) we

- devise algorithms for computing efficiently with algebras other than groups,
- investigate the border between problems in P and NP-complete problems.