Math 3140 - Assignment 12

Due April 29, 2024

(1) Let c be an integer and let $\sqrt{c} \in \mathbb{C}$ such that $(\sqrt{c})^2 = c$. Show that
$$\mathbb{Z}[c] := \{a + b\sqrt{c} \ : \ a, b \in \mathbb{Z}\}$$
is a subring of the field $(\mathbb{C}, +, \cdot)$.

Is $\mathbb{Z}[c]$ an integral domain? Is $\mathbb{Z}[c]$ a subfield of $\mathbb{C}$?

$\mathbb{Z}[i]$ for $i^2 = -1$ is called the ring of *Gaussian integers*.

(2) Let R be a finite commutative ring with 1. Show that every $a \in R \setminus \{0\}$ is either a unit or a zero divisor.

Find an infinite R for which this is not true.

Hint: Consider whether $aR = R$ or not.

(3) (a) Show that if an ideal I of a ring R contains a unit, then $I = R$.

(b) Conclude that the only ideals of a field F are 0 and F.

(4) Let R be a commutative ring with 1, let $a_1, \ldots, a_n \in R$.

(a) Show that
$$(a_1, \ldots, a_n) := \{r_1 a_1 + \cdots + r_n a_n \ : \ r_1, \ldots, r_n \in R\}$$
is an ideal of R.

(b) Show that any ideal I of R that contains $a_1, \ldots, a_n$ also contains $(a_1, \ldots, a_n)$. Thus $(a_1, \ldots, a_n)$ is the smallest ideal containing $a_1, \ldots, a_n$.

(5) Let I, J be ideals in a ring R. Show that their sum
$$I + J := \{i + j \ : \ i \in I, j \in J\}$$
is an ideal of R.

Describe the sum of the ideals (12) and (18) in $\mathbb{Z}$ as simply as possible.

(6) Show that the following pairs of rings are not isomorphic.

(a) $2\mathbb{Z}$ and $3\mathbb{Z}$

(b) $\mathbb{R} \times \mathbb{R}$ and $\mathbb{C}$

(c) $\mathbb{R}$ and $\mathbb{C}$

(7) Show that $\mathbb{R}[x]/(f)$ for $f = x^2 + 1$ is isomorphic to the field of complex numbers $\mathbb{C}$.

Hint: Note that elements in the quotient ring are of the form $a + bx + (f)$. What are the sum and the product of two such elements?