Math 3140 - Assignment 11

Due April 17, 2024

(1) Show every finite abelian group is the direct product of its Sylow subgroups.

Solution. Consider as example $G = \mathbb{Z}_4 \times \mathbb{Z}_9 \times Z_3$. The Sylow 2-subgroup of G is $\mathbb{Z}_4 \times 0 \times 0$. The Sylow 3-subgroup is $0 \times \mathbb{Z}_9 \times Z_3$.

By the Fundamental Theorem of Finite Abelian Groups every finite abelian group is isomorphic to some

$$\prod_{i=1}^{\ell} \underbrace{\mathbb{Z}_{p_1^{e_{11}}} \times \mathbb{Z}_{p_1^{e_{12}}} \times \cdots \times \mathbb{Z}_{p_1^{e_{1n_i}}}}_{P_i}$$

for distinct primes $p_1, \ldots, p_\ell$. Here P_i is isomorphic to the maximal subgroup of p_i-power order, hence the unique Sylow p_i-subgroup. $\square$

(2) Show that all Sylow p-subgroups of a finite group G for a prime p are isomorphic.

Solution. Let P_1, P_2 be Sylow p-subgroups of G. By Sylow's Second Thm they are conjugate, that is, there exists $g \in G$ such that $gP_1g^{-1} = P_2$. So $\varphi\colon P_1 \to P_2$, $x \mapsto gxg^{-1}$, is an isomorphism. $\square$

(3) (a) Find a Sylow 2-subgroup of S_4. Show that it is isomorphic to D_8. How many Sylow 2-subgroups are there?
(b) Find all Sylow 3-subgroup of S_4.
(c) Find all Sylow 5-subgroups of S_4

Solution. $|S_4| = 8 \cdot 3$.

(a) By Sylow's Third Thm S_4 has either 1 or 3 Sylow 2-subgroups of order 8. Recall that D_8 acts as permutations on the corners of a square, hence $D_8 \cong \langle (1234), (12)(34) \rangle =: P$ and the latter is a Sylow 2-subgroup of S_4. Can it be the only one?

Note that $g = (1324)$ is another 4-cycle in S_4, which is not contained in P. By Sylow's Second Thm g must be contained in some Sylow 2-subgroup. So $n_2 > 1$ and $n_2 = 3$.

(b) By Sylow's Third Thm S_4 has either 1 or 4 Sylow 3-subgroups isomorphic to $\mathbb{Z}_3$, each containing two 3-cycles. Recall that S_4 contains exactly $\frac{4!}{3} = 8$ cycles of length 3. Each

3-cycle is in exactly one Sylow 3-subgroup. As for A_4, there are 4 Sylow 3-subgroups for S_4, namely

$$\langle(123)\rangle,\ \langle(124)\rangle,\ \langle(134)\rangle,\ \langle(234)\rangle.$$

(c) The Sylow 5-subgroups of S_4 is trivial 1. $\square$

(4) Let $n \in \mathbb{N}$ be odd.
  (a) Give a Sylow 2-subgroup of D_{2n}. What is it isomorphic to? How many are there?
  (b) Let p be an odd prime. What are the Sylow p-subgroups of D_{2n}?
  (c) Are any of the Sylow subgroups of D_{2n} normal?

Solution. Let D_{2n} be generated by a rotation a of order n and a reflection b.

(a) Since $|D_{2n}| = 2n$ and n is odd, every Sylow 2-subgroup has size 2 (hence they are cyclic, isomorphic to $\mathbb{Z}_2$). Every element of order 2 in D_{2n} generates a Sylow 2-subgroup. Since n is odd, the element of order 2 are exactly the n reflections. Hence there are n Sylow 2-subgroups. They are not normal by Sylow's Second Theorem (All Sylow p-subgroups are conjugate).

(b) Let p an odd prime, and p^m maximal such that $p^m|2n$. Then $p^m|n$. Note that $\langle a\rangle$ contains an element of order p^m, namely a^{n/p^m}. Hence $\langle a^{n/p^m}\rangle$ is a Sylow p-subgroup of D_{2n}. Note that $\langle a^{n/p^m}\rangle$ is normal in D_{2n}. Hence it is the unique Sylow p-subgroup by Sylow's Second Theorem. $\square$

(5) For every prime p give a Sylow p-subgroup of A_5. Can you determine how many there are for each p? Are any of them normal?

Hint: Recall the number of permutations of a certain cycle structure.

Solution. $|A_5| = 60 = 4 \cdot 3 \cdot 5$. If $p \neq 2, 3, 5$, then the Sylow p-subgroups of A_5 are trivial.

$p = 5$: The Sylow 5-subgroup has order 5 and is cyclic. We find an element of order 5, e.g., the 5-cycle $(1\ 2\ 3\ 4\ 5)$. Note this is an odd cycle, hence in A_5. Then $\langle(1\ 2\ 3\ 4\ 5)\rangle$ is a Sylow 5-subgroup.

The number of Sylow 5-subgroup n_5 satisfies $n_5|\frac{60}{5}$ and $n_5 \equiv 1 \mod 5$ by the Third Sylow Theorem. Now $n_5|12$ and $n_5 \equiv 1 \mod 5$ yield options $n_5 = 1$ or $n_5 = 6$. Note that $\langle(1\ 2\ 3\ 5\ 4)\rangle$ is another Sylow 5-subgroup. So $n_5 \neq 1$ but $n_5 = 6$.

$p = 3$: The Sylow 3-subgroup has order 3 and is cyclic. We find an element of order 3, e.g., the 3-cycle $(1\ 2\ 3)$. Note this is an odd cycle, hence in A_5. Then $\langle(1\ 2\ 3)\rangle$ is a Sylow 3-subgroup.

The number of Sylow 3-subgroup n_3 satisfies $n_3|\frac{60}{3}$ and $n_3 \equiv 1$ mod 3 by the Third Sylow Theorem. Now $n_3|20$ and $n_3 \equiv 1$ mod 3 yield options $n_3 = 1, 4$ or 10. Note that A_5 contains $\frac{|S_5|}{|C_{S_5}((1\ 2\ 3))|} = \frac{120}{3\cdot 2} = 20$ 3-cycles. Each Sylow 3-subgroup contains 2 3-cycles and the identity. So the 20 3-cycles can be divided into 10 Sylow 3-subgroups.

$p = 2$: The Sylow 2-subgroup has order 4 and cannot be cyclic because A_5 contains no 4-cycle (these are odd permutation). So any Sylow 2-subgroup is not isomorphic to $\mathbb{Z}_4$ but to $\mathbb{Z}_2 \times \mathbb{Z}_2$. We look for 2 commuting elements of order 2 in A_5, e.g. $\langle(1\ 2)(3\ 4), (1\ 3)(2\ 4)\rangle$ is a Sylow 2-subgroup.

The number of Sylow 2-subgroup n_2 satisfies $n_2|\frac{60}{4}$ and $n_2 \equiv 1$ mod 2 by the Third Sylow Theorem. Now $n_2|15$ and $n_2 \equiv 1$ mod 2 yield options $n_2 = 1, 3, 5$ or 15. Note that A_5 contains $\frac{|S_5|}{|C_{S_5}((1\ 2)(3\ 4))|} = \frac{120}{2^2\cdot 2} = 15$ elements conjugate to $(1\ 2)(3\ 4)$. Since the only elements in A_5 that commute with $(1\ 2)(3\ 4)$ are in $\langle(1\ 2)(3\ 4), (1\ 3)(2\ 4)\rangle$, we have that $(1\ 2)(3\ 4)$ is contained in exactly one Sylow 2-subgroup. It follows that every conjugate $g(1\ 2)(3\ 4)g^{-1}$ for $g \in A_5$ is contained in exactly one Sylow 2-subgroup as well. Each Sylow 2-subgroup contains 3 elements of order 2 and the identity. So the 15 conjugates can be divided into 5 Sylow 2-subgroups.

Since none of the non-trivial Sylow subgroups appear exactly once, none of them are normal. In fact A_5 is simple. □

(6) (a) Show that every group of order 56 has a proper non-trivial normal subgroup.
(b) Show that every group of order 175 is abelian.

Hint: Determine the numbers of Sylow subgroups.

Solution. (a) Let $|G| = 56 = 8 \cdot 7$. By Sylow's Third Thm G has either 1 or 8 Sylow 7-subgroups each isomorphic to $\mathbb{Z}_7$.

Case $n_7 = 1$: Then G has a unique and normal Sylow 7-subgroup.

Case $n_7 = 8$: Any 2 distinct Sylow 7-subgroups intersect only in the trivial subgroup. So G has $8 \cdot 6 = 48$ elements of order 7. Hence G has only $56 - 48 = 8$ elements whose order is not 7. But G has a Sylow 2-subgroup P of order 8. So these 8 remaining elements are the elements of P and there is only one Sylow 2-subgroup, hence normal.

(b) Let $|G| = 175 = 7 \cdot 25$. By Sylow's Third Thm G has 1 Sylow 7-subgroup P isomorphic to $\mathbb{Z}_7$. Similar G has 1 Sylow 5-subgroup Q isomorphic to $\mathbb{Z}_{25}$ or $\mathbb{Z}_5^2$. Since P, Q are normal in G, their product PQ is a subgroup of G. But PQ contains both P and Q. So $|PQ|$ is a multiple of $7 \cdot 25$ and $PQ = G$. Clearly $P \cap Q = 1$ by Lagrange's Thm. So $G \cong P \times Q$ is a direct product of abelian groups, hence abelian. $\square$

(7) (a) How many groups of size 21 are there up to isomorphism? What do they look like?
(b) How many groups of size 33?

Solution. $21 = 3 \cdot 7$ and $3|7-1$. So there are exactly two groups of size 21 up to isomorphism: $\mathbb{Z}_{21}$ and the non-abelian group $\langle a, b : \ a^7 = 1, b^3 = 1, bab^{-1} = a^2 \rangle$.

$33 = 3 \cdot 11$ and $3 \not| 11 - 1$. So there is exactly one group of size 33 up to isomorphism: $\mathbb{Z}_{33}$. $\square$