Math 3140 - Assignment 10

Due April 3, 2024

This assignment is a set of practice problems for the midterm exam on April 8.

(1) Let G, H be groups. Show that

$$Z(G \times H) = Z(G) \times Z(H).$$

Solution. $(g,h) \in Z(G \times H)$
$\Leftrightarrow \forall x \in G, y \in H:\ (g,h)(x,y) = (x,y)(g,h)$
$\Leftrightarrow \forall x \in G, y \in H:\ gx = xg, hy = yh$
$\Leftrightarrow g \in Z(G), h \in Z(H)$
$\Leftrightarrow (g,h) \in Z(G) \times Z(H)$ □

(2) (a) Describe all isomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_4 \times \mathbb{Z}_3$. How many are there?
(b) Show that every homomorphism from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$ is of the form $(x,y) \mapsto ax + by$ for some integers a, b.

Solution. (a) Every isomorphism has to map the generator 1 of $\mathbb{Z}_{12}$ to an element of order 12. So there are 4 isomorphisms determined by
$1 \mapsto (1,1)$
$1 \mapsto (1,2)$
$1 \mapsto (3,1)$
$1 \mapsto (3,2)$
(b) Clearly every map $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z},\ (x,y) \mapsto ax + by$, is a homomorphism. It remains to show that every homomorphism φ is of this form. Assume $\varphi(1,0) = a$ and $\varphi(0,1) = b$. Let $(x,y) \in \mathbb{Z}^2$. Then

$$\varphi(x,y) = \varphi((x,0) + (0,y)) = \varphi(x,0) + \varphi(0,y)$$
$$\varphi(x(1,0)) + \varphi(y(0,1)) = x\varphi(1,0) + y\varphi(0,1) = ax + by$$

□

(3) Let N be a normal subgroup of G such that G/N is abelian. Show that $x^{-1}y^{-1}xy \in N$ for all $x, y \in G$.

The expression $x^{-1}y^{-1}xy$ is called the *commutator* of x and y and denoted by $[x,y]$.

Solution. Let $x, y \in G$. Then

$$xNyN = yNxN \ \Leftrightarrow xyN = yxN \ \Leftrightarrow x^{-1}yxyN$$

The forward implications show (3), the backwards implications show (4b). □

(4) Let G' be the subgroup of G that is generated by the set of all commutators $\{[x,y] : x,y \in G\}$. Then G' is called the *commutator subgroup* or *derived subgroup* of G.

(a) Show that G' is normal in G.

Hint: Show that any conjugate of a commutator is a commutator.

(b) Show that G/G' is abelian.

Solution. (a) Let $x, y, g \in G$. Then

$$[x,y]^g = gx^{-1}y^{-1}xyg^{-1} = gx^{-1}g^{-1}gy^{-1}g^{-1}gxg^{-1}gyg^{-1} = [x^g, y^g].$$

Hence every conjugate of a generator of G' is in G' again. So every conjugate of any element of G' is contained in G'. Thus G' is normal in G. □

(5) By (3) and (4) the commutator subgroup G' is the smallest normal subgroup N of G such that G/N is abelian.

Use this and what you know about normal subgroups of the following groups to determine G' for

(a) G abelian, (b) S_3, (c) D_8 (d) A_4.

You do not need to compute any commutators $[x, y]$ for this.

Solution. (a) G is abelian iff $G' = 1$.

(b) S_3/A_3 is abelian. The only normal subgroup smaller than A_3 is the trivial subgroup 1 and $S_3/1 \cong S_3$ is not abelian. So $S_3' = A_3$.

(c) Recall $D_8 = \langle a, b : a^4 = 1, b^2 = 1, bab^{-1} = a^{-1}$. $D_8/\langle a^2\rangle$ has size 4 and is abelian. The only normal subgroup smaller than $\langle a^2\rangle$ is the trivial subgroup 1 and $D_8/1 \cong D_8$ is not abelian. So $D_8' = \langle a^2\rangle$.

(d) $A_4/V_4 \cong \mathbb{Z}_3$ is abelian and this is the largest abelian quotient of A_4. So $A_4' = V_4$. □

(6) (a) Find all abelian groups of order 360 up to isomorphism.

(b) Which of these groups have exactly 3 elements of order 2?

Solution. (a) $360 = 2^3 \cdot 3^2 \cdot 5$. Partitioning the exponents of the prime powers yields 6 non-isomorphic abelian groups of order 360:

$\mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
$\mathbb{Z}_8 \times (Z_3)^2 \times \mathbb{Z}_5$
$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
$\mathbb{Z}_4 \times \mathbb{Z}_2 \times (\mathbb{Z}_3)^2 \times \mathbb{Z}_5$
$(\mathbb{Z}_2)^3 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
$(\mathbb{Z}_2)^3 \times (\mathbb{Z}_3)^2 \times \mathbb{Z}_5$

(b) Since $\mathbb{Z}_8$ has exactly 1 element of order 2, so do the first 2 groups.

Since $\mathbb{Z}_4 \times \mathbb{Z}_2$ has exactly 3 elements of order 2 (namely $(0,1), (2,0), (2,1)$) so do the middle 2 groups.

Since $\mathbb{Z}_2^3$ has exactly 7 elements of order 2 (namely all but $(0,0,0)$) so do the last 2 groups.

(7) (a) How many colorings are there of the faces of a cube in 2 colors up to rotational symmetry? (Two colorings are considered equivalent when one can be obtained from the other by rotating the cube.)

(b) How many ways can you label the faces of a die $1, \dots, 6$ up to rotational symmetry?

Hint: recall the rotation group of the cube from [1, Thm 7.4].

Solution. There are 24 rotational symmetries of the cube in total:

	g	multiplicity	for (a): Fix(g)	for (b): Fix(g)
identity	$()$	1	2^6	$6!$
$90°, 270°$ around center of face	$(1\ 2\ 3\ 4)$	6	2^3	0
$180°$ around center of face	$(1\ 3)(2\ 4)$	3	2^4	0
$120°, 240°$ around diagonal	$(1\ 2\ 6)(3\ 4\ 5)$	8	2^2	0
$180°$ around center of sides	$(1\ 4)(2\ 3)(5\ 6)$	6	2^3	0

(a) We get $\frac{1}{24}(64 + 6*8 + 3*16 + 8*4 + 6*8) = 10$ colorings in 2 colors up to rotational symmetry.

(b) We get $\frac{720}{24} = 30$ labellings of the faces of a die up to rotational symmetry.

References

[1] Joseph A. Gallian. Contemporary Abstract Algebra. Houghton Mifflin Company, sixth edition, 2006.