# Math 3140 - Assignment 10

Due April 3, 2024

This assignment is a set of practice problems for the midterm exam on April 8.

(1) Let $G, H$ be groups. Show that

$$Z(G \times H) = Z(G) \times Z(H).$$

(2) (a) Describe all isomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_4 \times \mathbb{Z}_3$. How many are there?
  (b) Show that every homomorphism from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$ is of the form $(x, y) \mapsto ax + by$ for some integers $a, b$.

(3) Let $N$ be a normal subgroup of $G$ such that $G/N$ is abelian. Show that $x^{-1}y^{-1}xy \in N$ for all $x, y \in G$.
  The expression $x^{-1}y^{-1}xy$ is called the *commutator* of $x$ and $y$ and denoted by $[x, y]$.

(4) Let $G'$ be the subgroup of $G$ that is generated by the set of all commutators $\{[x, y] : x, y \in G\}$. Then $G'$ is called the *commutator subgroup* or *derived subgroup* of $G$.
  (a) Show that $G'$ is normal in $G$.
  Hint: Show that any conjugate of a commutator is a commutator.
  (b) Show that $G/G'$ is abelian.

(5) By (3) and (4) the commutator subgroup $G'$ is the smallest normal subgroup $N$ of $G$ such that $G/N$ is abelian.
  Use this and what you know about normal subgroups of the following groups to determine $G'$ for
  (a) $G$ abelian, (b) $S_3$, (c) $D_8$ (d) $A_4$.
  You do not need to compute any commutators $[x, y]$ for this.

(6) (a) Find all abelian groups of order 360 up to isomorphism.
  (b) Which of these groups have exactly 3 elements of order 2?

(7) (a) How many colorings are there of the faces of a cube in 2 colors up to rotational symmetry? (Two colorings are considered equivalent when one can be obtained from the other by rotating the cube.)
  (b) How many ways can you label the faces of a die $1, \ldots, 6$ up to rotational symmetry?
  Hint: recall the rotation group of the cube from [1, Thm 7.4].

## References

[1] Joseph A. Gallian. Contemporary Abstract Algebra. Houghton Mifflin Company, sixth edition, 2006.