Math 3140 - Assignment 9

Due March 20, 2024

(1) Let $n \in \mathbb{N}$. Show that sign: $S_n \to (\{-1,1\},\cdot)$ defined by

$$\operatorname{sign}(f) := \frac{\prod_{1\leq i<j\leq n}(f(j)-f(i))}{\prod_{1\leq i<j\leq n}(j-i)}$$

is a homomorphism.

Solution: Since $f \in S_n$ just permutates the elements in $\{1,\dots,n\}$, $\prod_{1\leq i<j\leq n}(j-i)$ and $\prod_{1\leq i<j\leq n}(f(j)-f(i))$ have the same absolute value. Hence sign$f$ is either 1 or -1.

To show that sign is a homomorphism, let $f, g \in S_n$ and consider

$$\begin{aligned}\operatorname{sign}(fg) &= \prod_{1\leq i<j\leq n} \frac{fg(j)-fg(i)}{j-i} \\ &= \prod_{1\leq i<j\leq n} \left(\frac{fg(j)-fg(i)}{g(j)-g(i)} \cdot \frac{g(j)-g(i)}{j-i}\right) \\ &= \prod_{1\leq i<j\leq n} \frac{fg(j)-fg(i)}{g(j)-g(i)} \cdot \underbrace{\prod_{1\leq i<j\leq n} \frac{g(j)-g(i)}{j-i}}_{\operatorname{sign}(g)}\end{aligned}$$

To see that the first factor in the product above is sign$f$, note that

$$\frac{fg(j)-fg(i)}{g(j)-g(i)} = \frac{fg(i)-fg(j)}{g(i)-g(j)}.$$

Hence we can reorder each factor in $\prod_{1\leq i<j\leq n}\frac{fg(j)-fg(i)}{g(j)-g(i)}$ such that the denominator is positive. Doing that and using that g just permutes the elements in $\{1,\dots,n\}$ we obtain

$$\prod_{1\leq i<j\leq n} \frac{fg(j)-fg(i)}{g(j)-g(i)} = \prod_{1\leq k<l\leq n} \frac{f(l)-f(k)}{l-k}.$$

Thus $\operatorname{sign}(fg) = \operatorname{sign}(f)\operatorname{sign}(g)$ follows. □

(2) Show $\operatorname{sign}(f) = -1$ for any transposition $f = (a\ b)$ in S_n.

Hint: Count the number of inversions of f, that is, the pairs $1 \leq x < y \leq n$ such that $f(x) > f(y)$. Recall from class that $\operatorname{sign}(f) = (-1)^{\text{number of inversions of } f}$.

Solution. Let $f = (k, l)$ be a transposition with $1 \leq k < l \leq n$. To compute sign$f$ we just need to count how many of the factors $f(j) - f(i)$ for $1 \leq i < j \leq n$ in the enumerator of the formula for sign$f$ are negative.

Note that f fixes all pairs $i < j$ if both i, j are distinct from k, l. Consequently $f(j) - f(i)$ is negative only if one of i or j are k or l, in particular if

- $i = k$ and $j \in \{k+1, \ldots, l-1\}$,
- $j = l$ and $i \in \{k+1, \ldots, l-1\}$, or
- $i = k, j = l$.

So in total there $2(l-k-1)+1$ instances where $f(j) - f(i)$ is negative. Since this number is odd, sign$f = -1$.

Alternatively: it is easy to see that $\text{sign}(1,2) = -1$. One can show that every transposition f in S_n is conjugate to $(1,2)$, that is, $f = g(1,2)g^{-1}$ for some $g \in S_n$. Since sign is a homomorphism into a commutative group, this implies

$$\text{sign}f = \text{sign}(g(1,2)g^{-1}) = \text{sign}g\text{sign}(1,2)(\text{sign}g)^{-1}$$

$$= \text{sign}g(\text{sign}g)^{-1}\text{sign}(1,2) = \text{sign}(1,2) = -1.$$

□

(3) When are two elements of S_n conjugate?

(a) Show that for any k-cycle $(a_1, a_2, \ldots, a_k) \in S_n$ and any $f \in S_n$, we have

$$f(a_1, a_2, \ldots, a_k)f^{-1} = (f(a_1), f(a_2), \ldots, f(a_k)).$$

(b) For any two k-cycles $(a_1, a_2, \ldots, a_k), (b_1, b_2, \ldots, b_k) \in S_n$ explicitly give $f \in S_n$, such that

$$f(a_1, a_2, \ldots, a_k)f^{-1} = (b_1, b_2, \ldots, b_k).$$

The *cycle structure* of a permutation g is the length of the cycles in the cycle decomposition of g (counted with multiplicity). For example $g = (1\ 2\ 3)(4\ 5)(6\ 7)$ has cycle structure $3, 2, 2$.

Deduce that two permutations $g, h \in S_n$ are conjugate iff they have the same cycle structure.

Solution.

(a) Let $c := (a_1, a_2, \ldots, a_k)$. To show that fcf^{-1} and $d := (f(a_1), f(a_2), \ldots, f(a_k))$ are the same functions, we check that $fcf^{-1}(x) = g(x)$ for all $x \in \{1, \ldots, n\}$.

Case 1, $x \in \{f(a_1), f(a_2), \dots, f(a_k)\}$: Suppose $x = f(a_i)$. Then

$$fcf^{-1}(x) = fcf^{-1}f(a_i) = fc(a_i) = \begin{cases} f(a_{i+1}) & \text{if } i < k \\ f(a_1) & \text{if } i = k \end{cases}$$

For $g(x)$ we get the same result.
Case 2, $x \in \{1, \dots, n\} \setminus \{f(a_1), f(a_2), \dots, f(a_k)\}$: Then $f^{-1}(x) \notin \{a_1, a_2, \dots, a_k\}$ and $f^{-1}(x)$ is fixed by c. So

$$fcf^{-1}(x) = ff^{-1}(x) = x$$

For $g(x)$ we get the same result.

(b) Define $f \in S_n$ by $f(a_1) := b_1, \dots, f(a_k) := b_k$ and as an arbitrary bijection from $\{1, \dots, n\} \setminus \{a_1, a_2, \dots, a_k\}$ to $\{1, \dots, n\} \setminus \{b_1, b_2, \dots, b_k\}$. □

(4) (a) How many different conjugacy classes are there in S_4?
(b) For $g = (1\ 2)(3\ 4)$ determine $C_{S_4}(g)$, the centralizer of g in S_4.
(c) How many elements in S_4 are conjugate to $(1\ 2)(3\ 4)$?

Hint: Use (3) and the Orbit-Stabilizer Theorem

Solution. (a) By (3) the number of conjugacy classes of S_4 is equal to the number of different cycle structures of elements in S_4, which is just the number of partitions of 4:

$$\begin{aligned} 4&, \text{ class of } (1234) \\ 3+1&, \text{ class of } (123) \\ 2+2&, \text{ class of } (12)(34) \\ 2+1+1&, \text{ class of } (12) \\ 1+1+1+1&, \text{ class of } () \end{aligned}$$

Hence there are 5 conjugacy classes.
(b) Let $f \in C_{S_4}(g)$. Then by (5)

$$g = fgf^{-1} = (f(1)\ f(2))(f(3)\ f(4))$$

We see that $\{f(1), f(2)\}$ is either $\{1, 2\}$ or $\{3, 4\}$. In particular $f(2)$ is uniquely determined by $f(1)$. Next $\{f(3), f(4)\}$ must be $\{1, 2, 3, 4\} \setminus \{f(1), f(2)\}$ and $f(4)$ is uniquely determined by $f(3)$.

So f is uniquely determined by $f(1)$ and $f(3)$. There are $4 * 2 = 8$ options for $f(1)$ and $f(3)$. Hence $|C_{S_4}(g)| = 8$.

To see the centralizing elements explicitly: $\langle(1\ 3\ 2\ 4),(1\ 3)\rangle$ is a group of size 8 that is contained in $C_{S_4}(g)$. Hence

$$C_{S_4}(g)=\langle(1\ 3\ 2\ 4),(1\ 3)\rangle.$$

□

(5) Which of the following are group actions? Check the properties. Are they transitive?
   (a) G on $X := G/H$ for a subgroup H of G by $g * xH := gxH$
   (b) G on $X := G$ by $g * x := g^{-1}xg$

   Solution.
   (a) Group action because 1 fixes all cosets xG and $(gh)xH = g(hxH)$ for all $g, h, x \in G$.
   There is just one orbit since for any $g \in G$, the coset H can be translated into gH. Hence the action is transitive.
   (b) In general not a group action since $(gh) * x = (gh)^{-1}xgh$ whereas $g * (h * x) = g^{-1}h^1xhg$.
   Note that conjugation becomes a group action when defining it by $g * x := gxg^{-1}$. □

(6) For $(G, \cdot)$ acting on a set X and $x, y \in X$, define $x \sim y$ if $\exists g \in G \colon y = gx$. Show:
   (a) $\sim$ is an equivalence relation on X.
   (b) The orbit $Gx := \{gx : g \in G\}$ is the equivalence class of x with respect to $\sim$.

   Solution.
   (a) Reflexive because $x = 1x$ for all $x \in X$.
   Symmetric because $y = gx$ implies $x = g^{-1}y$.
   Transitive because $y = gx$ and $z = hy$ imply $z = hgx$.
   (b) follows since $\{y : y = gx \text{ for some } g \in G\} = \{gx : g \in G\}$. □

(7) (a) How many distinct necklaces can be made with 2 red, 2 blue and 2 green beads?
   (b) How many distinct necklaces can be made with 6 beads of (at most) 3 different colors?

   Solution: For (a) we have D_{12} act on the set X of all possible arrangements of 2 red, 2 blue and 2 green beads, that is $|X| = \binom{6}{2}\binom{4}{2}\binom{2}{2} = 90$.

   For (b) we have D_{12} act on the set Y of all possible colorings of 6 vertices in red, blue and green, that is $|Y| = 3^6$.

   We list how often each cycle structure occurs in D_{12} and how many fixed points the corresponding permutation has in X, Y, respectively. The first 4 rows correspond to rotations, the last 2 to reflections of the hexagon. Note that in every cycle the fixed elements have to have the same color.

multiplicity	$g \in D_{12}$	$\|\text{fix}(g)\|$ in X	$\|\text{fix}(g)\|$ in Y
1	()	90	3^6
2	(1 2 3 4 5 6)	0	3^1
2	(1 3 5)(2 4 6)	0	3^2
1	(1 4)(2 5)(3 6)	3!	3^3
3	(1 2)(3 6)(2 5)	3!	3^3
3	(2 6)(3 5)(1)(4)	3!	3^4

By the Burnside-Frobenius Lemma we get

$$\frac{1}{12}(90 + 2 \cdot 0 + 2 \cdot 0 + 1 \cdot 6 + 3 \cdot 6 + 3 \cdot 6) = 11$$

orbits for (a). Similarly 92 orbits for (b). □

(8) Recall that the rotation group of a regular tetrahedron acts on the 4 vertices (equivalently the 4 faces) like A_4.
  (a) In how many ways can the faces of a regular tetrahedron be colored with 4 colors so that every color occurs exactly once?
  Try to draw the colorings and explain your result geometrically.
  (b) In how many ways can the faces of a regular tetrahedron be colored with 4 colors without any restrictions?

Solution: For (a) we have A_4 act on the set X of all colorings X in 4 distinct colors, that is $|X| = 4!$.

For (b) we have A_4 act on the set Y of all possible colorings in 4 colors, that is $|Y| = 4^4$.

We list how often each cycle structure occurs in A_4 and how many fixed points the corresponding permutation has in X, Y, respectively. In every cycle the fixed elements have to have the same color.

multiplicity	$g \in A_4$	$\|\text{fix}(g)\|$ in X	$\|\text{fix}(g)\|$ in Y
1	()	4!	4^4
3	(1 2)(3 4)	0	4^2
8	(1 2 3)	0	4^2

By the Burnside-Frobenius Lemma we get

$$\frac{1}{12}(24 + 2 \cdot 0 + 3 \cdot 0 + 8 \cdot 0) = 2$$

orbits for (a).

For (a) note that coloring the faces numbered $1, 2, 3, 4$ of the tetrahedron in colors called $1, 2, 3, 4$ can be considered as a permutation of $1, 2, 3, 4$. The action of A_4 maps "even" colorings to "even" colorings as well as "odd" colorings to "odd" colorings. But it cannot change an "even" coloring to an "odd" one.

Geometrically A_4 cannot translate a coloring of the tetrahedron to the coloring of the mirrored tetrahedron. So there are exactly 2 orbits in (a).

For (b) we get

$$\frac{1}{12}(4^4 + 11 \cdot 4^2) = 36$$

orbits. □