Math 3140 - Assignment 9

Due March 20, 2024

(1) Let $n \in \mathbb{N}$. Show that $\text{sign}\colon S_n \to (\{-1, 1\}, \cdot)$ defined by

$$\text{sign}(f) := \frac{\prod_{1 \le i < j \le n}(f(j) - f(i))}{\prod_{1 \le i < j \le n}(j - i)}$$

for $f \in S_n$ is a homomorphism.

(2) Show $\text{sign}(f) = -1$ for any transposition $f = (a\ b)$ in S_n.
Hint: Count the inversions of f, that is, the pairs (x, y) such that $1 \le x < y \le n$ but $f(x) > f(y)$. Recall from class that

$$\text{sign}(f) = (-1)^{\text{number of inversions of } f}.$$

(3) When are two elements of S_n conjugate?
   (a) Show that for any k-cycle $(a_1, a_2, \ldots, a_k) \in S_n$ and any $f \in S_n$, we have
   $$f(a_1, a_2, \ldots, a_k)f^{-1} = (f(a_1), f(a_2), \ldots, f(a_k)).$$
   (b) For any two k-cycles $(a_1, a_2, \ldots, a_k), (b_1, b_2, \ldots, b_k) \in S_n$ explicitly give $f \in S_n$, such that
   $$f(a_1, a_2, \ldots, a_k)f^{-1} = (b_1, b_2, \ldots, b_k).$$

The *cycle structure* of a permutation g is the length of the cycles in the cycle decomposition of g (counted with multiplicity). For example $g = (1\ 2\ 3)(4\ 5)(6\ 7)$ has cycle structure $3, 2, 2$.

Deduce that two permutations $g, h \in S_n$ are conjugate iff they have the same cycle structure.

(4) (a) How many different conjugacy classes are there in S_4?
   (b) For $g = (1\ 2)(3\ 4)$ determine $C_{S_4}(g)$, the centralizer of g in S_4.
   (c) How many elements in S_4 are conjugate to $(1\ 2)(3\ 4)$?

Hint: Use (3) and the Orbit-Stabilizer Theorem

(5) Which of the following are group actions? Check the properties. Are they transitive?
   (a) G on $X := G/H$ for a subgroup H of G by $g * xH := gxH$
   (b) G on $X := G$ by $g * x := g^{-1}xg$

(6) For $(G, \cdot)$ acting on a set X and $x, y \in X$, define $x \sim y$ if $\exists g \in G\colon y = gx$. Show:
   (a) $\sim$ is an equivalence relation on X.
   (b) The orbit $Gx := \{gx\ :\ g \in G\}$ is the equivalence class of x with respect to $\sim$.

(7) (a) How many distinct necklaces can be made with 2 red, 2 blue and 2 green beads?
    (b) How many distinct necklaces can be made with 6 beads of (at most) 3 different colors?

(8) Recall that the rotation group of a regular tetrahedron acts on the 4 vertices (equivalently the 4 faces numbered $1, 2, 3, 4$) like A_4 does.
    (a) In how many ways can the faces of a regular tetrahedron be colored with 4 colors so that every color occurs exactly once?
    (b) In how many ways can the faces of a regular tetrahedron be colored with 4 colors without any restrictions?