Math 3140 - Assignment 8

Due March 13, 2024

(1) Determine all homomorphisms from G to H and their kernels. Which are surjective?

(a) $G = \mathbb{Z}_4,\ H = \mathbb{Z}_2 \times \mathbb{Z}_2$
(b) $G = H = \mathbb{Z}_n$
(c) $G = S_4,\ H = \mathbb{Z}_2$
(d) $G = \mathbb{Z},\ H = S_3$

Hint: Consider where the generators of G can be mapped under homomorphisms. Use the First Isomorphism Theorem.

Solution. Recall that a homomorphism $\varphi\colon G \to H$ is uniquely determined by where φ sends the generators of G.

(a) $\varphi_{00}\colon \mathbb{Z}_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2,\ x \mapsto (0,0)$, has kernel $\mathbb{Z}_4$.
$\varphi_{10}\colon x \mapsto ([x]_2, 0)$,
$\varphi_{10}\colon x \mapsto (0, [x]_2)$,
$\varphi_{11}\colon x \mapsto ([x]_2, [x]_2)$, all have kernel $\mathbb{Z}_2$.
No homomorphism is surjective.

(b) Every homomorphism is of the form $\varphi_a\colon \mathbb{Z}_n \to \mathbb{Z}_n,\ x \mapsto ax$, for $a \in \mathbb{Z}_n$,
$\ker \varphi_a = \frac{n}{\gcd(a,n)}\mathbb{Z}_n$ and φ_a is surjective (bijective) iff $\gcd(a,n) = 1$.

(c) Let $\varphi\colon S_4 \to \mathbb{Z}_2$ be a homomorphism. Then $\varphi(S_4) \leq \mathbb{Z}_2$.
Case $\varphi(S_4) = 0$, not surjective: Then $\ker \varphi(S_4) = S_4$ and φ is the constant 0-map.
Case $\varphi(S_4) = \mathbb{Z}_2$, surjective: Then $\ker \varphi(S_4)$ is a normal subgroup of index 2 in S_4. The only such subgroup is A_4. So $\ker \varphi(S_4) = A_4$ and $\varphi(A_4) = 0, \varphi((12)A_4) = 1$.

(d) Every homomorphism is of the form $\varphi_a\colon \mathbb{Z} \to S_3$ defined by $\varphi(1) = a \in S_3$. None of them are surjective since S_3 is not cyclic. If $|a| = n$, then $\ker \varphi_a = n\mathbb{Z}$. □

(2) For a subgroup H and a normal subgroup N of G show that

$$HN := \{hn \ :\ h \in H, n \in N\}$$

is a subgroup of G.

Solution.

- $1 \in HN \neq \emptyset$.
- Let $h_1, h_2 \in H, n_1, n_2 \in N$. Since N is normal,

$$(h_1 n_1)(h_2 n_2) = h_1 h_2 (h_2^{-1} n_2 h_2) n_2 \in HN.$$

- Let $h \in H, n \in N$. Then $(hn)^{-1} = n^{-1}h^{-1} = h^{-1}(hn^{-1}h^{-1}) \in HN$. □

(3) **Characterization of direct products:**
(a) Let $G = K \times N$ be an external direct product. Show that $K \times 1$ and $1 \times N$ are normal subgroups of G such that

$$(K \times 1) \cap (1 \times N) = 1 \quad \text{and} \quad (K \times 1)(1 \times N) = G.$$

(b) Let G be a group with normal subgroups K, N such that

$$K \cap N = 1 \quad \text{and} \quad KN = G.$$

Then G is called an **internal direct product** of K and N. Show that

$$G \cong K \times N.$$

Hint: Show that $\varphi\colon K \times N \to G, (k, n) \mapsto kn$, is an isomorphism.

Solution.
(a) $K \times 1 \trianglelefteq G$ since $(x, y)^{-1}\ K \times 1\ (x, y) = x^{-1}Kx \times 1 = K \times 1$ for all $(x, y) \in G$. Similarly $1 \times N \trianglelefteq G$. $(K \times 1) \cap (1 \times N) = 1$ and $(K \times 1)(1 \times N) = G$ are clear.
(b) $\varphi\colon K \times N \to G, (k, n) \mapsto kn$, is
- surjective by the assumption $KN = G$;
- injective: for $(x_1, y_1), (x_2, y_2) \in K \times N$, $x_1y_1 = x_2y_2$ yields $x_2^{-1}x_1 = y_2y_1^{-1} \in K \cap N = 1$;
- a homomorphism: for $(x_1, y_1), (x_2, y_2) \in K \times N$,

$$\varphi((x_1, y_1)(x_2, y_2)) = x_1x_2y_1y_2$$
$$\varphi(x_1, y_1)\varphi(x_2, y_2) = x_1y_1x_2y_2$$

are equal since any $x \in K$ commutes with any $y \in N$:

$$x^{-1}\underbrace{y^{-1}xy}_{\in K} = \underbrace{x^{-1}y^{-1}x}_{\in N}\,y \in K \cap N = 1.$$

(4) **Correspondence Theorem between normal subgroups:**
Let $\varphi\colon G \to H$ be an onto homomorphism. Show
(a) If B is normal in H, then $\varphi^{-1}(B)$ is normal in G.
(b) If A is normal in G, then $\varphi(A)$ is normal in H.

Solution. (a) Assume B is normal in H. Let $x \in \varphi^{-1}(B)$ and $g \in G$. To show $gxg^{-1} \in \varphi^{-1}(B)$ consider

$$\varphi(gxg^{-1}) = \varphi(g)\varphi(x)\varphi(g)^{-1}$$

which is in B because $\varphi(x)$ is in B and B is normal. Hence $\varphi^{-1}(B)$ is normal.
(b) is similar.

(5) Are the following groups isomorphic?
  (a) $\mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
  (b) $\mathbb{Z}_{10} \times \mathbb{Z}_{12} \times \mathbb{Z}_6$ and $\mathbb{Z}_{60} \times \mathbb{Z}_6 \times \mathbb{Z}_2$
  (c) $\mathbb{Z}_{10} \times \mathbb{Z}_{12} \times \mathbb{Z}_6$ and $\mathbb{Z}_{15} \times \mathbb{Z}_4 \times \mathbb{Z}_{12}$

Solution (a) No, since the first has 3 elements of order 2 and the second 7 (or by the Fundamental Theorem of Finite Abelian Groups).

(b) Yes, splitting into prime power factors shows that both are isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$.

(c) No. The first splits into factors $\mathbb{Z}_4 \times \mathbb{Z}_2^2 \times \mathbb{Z}_3^2 \times \mathbb{Z}_5$, the second into $\mathbb{Z}_4^2 \times \mathbb{Z}_3^2 \times \mathbb{Z}_5$. By the Fundamental Theorem of Finite Abelian Groups these groups are not isomorphic.

(6) How many abelian groups up to isomorphism are there of order
  (a) 6,
  (b) 15,
  (c) 30,
  (d) pq for distinct primes p, q
  (e) n where n is a product of pairwise distinct primes?

Solution. A product of pairwise distinct primes means $n = p_1 p_2 \dots p_k$ for distinct primes $p_1, \dots, p_k$. By the Fundamental Theorem of Finitely Generated Abelian Groups every abelian group of that order is isomorphic to

$$\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \dots \mathbb{Z}_{p_k},$$

which is isomorphic to $\mathbb{Z}_n$ since these factors all have coprime orders.

(7) (a) Find all abelian groups of order 180 up to isomorphism.
  (b) For a prime p prime, find all abelian groups of order p^5 up to isomorphism.

Solution. (a) Since $180 = 2^2 * 3^2 * 5$ and the exponent 2 has only 2 partitions, we have 4 options:
$\mathbb{Z}_2^2 \times \mathbb{Z}_3^2 \times \mathbb{Z}_5$
$\mathbb{Z}_4 \times \mathbb{Z}_3^2 \times \mathbb{Z}_5$
$\mathbb{Z}_2^2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
$\mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5$

(b)

partitions of 5	groups
5	$\mathbb{Z}_{p^5}$
$4+1$	$\mathbb{Z}_{p^4} \times \mathbb{Z}_p$
$3+2$	$\mathbb{Z}_{p^3} \times \mathbb{Z}_{p^2}$
$3+1+1$	$\mathbb{Z}_{p^3} \times \mathbb{Z}_p \times \mathbb{Z}_p$
$2+2+1$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2} \times \mathbb{Z}_p$
$2+1+1+1$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$
$1+1+1+1+1$	$\mathbb{Z}_p^5$