Math 3140 - Assignment 8

Due March 13, 2024

(1) Determine all homomorphisms from G to H and their kernels. Which are surjective?
   (a) $G = \mathbb{Z}_4,\ H = \mathbb{Z}_2 \times \mathbb{Z}_2$
   (b) $G = H = \mathbb{Z}_n$
   (c) $G = S_4,\ H = \mathbb{Z}_2$
   (d) $G = \mathbb{Z},\ H = S_3$

   Hint: Consider where the generators of G can be mapped under homomorphisms. Use the First Isomorphism Theorem.

(2) For a subgroup H and a normal subgroup N of G show that

$$HN := \{hn \ : \ h \in H, n \in N\}$$

   is a subgroup of G.

(3) **Characterization of direct products:**
   (a) Let $G = K \times N$ be an external direct product. Show that $K \times 1$ and $1 \times N$ are normal subgroups of G such that

$$(K \times 1) \cap (1 \times N) = 1 \quad \text{and} \quad (K \times 1)(1 \times N) = G.$$

   (b) Let G be a group with normal subgroups K, N such that

$$K \cap N = 1 \quad \text{and} \quad KN = G.$$

   Then G is called an **internal direct product** of K and N. Show that

$$G \cong K \times N.$$

   Hint: Show that $\varphi\colon K \times N \to G, (k,n) \mapsto kn$, is an isomorphism.

(4) **Correspondence Theorem between normal subgroups:** Let $\varphi\colon G \to H$ be an onto homomorphism. Show
   (a) If B is normal in H, then $\varphi^{-1}(B)$ is normal in G.
   (b) If A is normal in G, then $\varphi(A)$ is normal in H.

(5) Are the following groups isomorphic?
   (a) $\mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
   (b) $\mathbb{Z}_{10} \times \mathbb{Z}_{12} \times \mathbb{Z}_6$ and $\mathbb{Z}_{60} \times \mathbb{Z}_6 \times \mathbb{Z}_2$
   (c) $\mathbb{Z}_{10} \times \mathbb{Z}_{12} \times \mathbb{Z}_6$ and $\mathbb{Z}_{15} \times \mathbb{Z}_4 \times \mathbb{Z}_{12}$

(6) How many abelian groups up to isomorphism are there of order
   (a) 6,
   (b) 15,
   (c) 30,
   (d) pq for distinct primes p, q

   (e) n where n is a product of pairwise distinct primes?

(7) (a) Find all abelian groups of order 180 up to isomorphism.
   (b) For a prime p prime, find all abelian groups are there of order p^5 up to isomorphism.