Math 3140 - Assignment 7

Due March 6, 2024

(1) For a group G and $x, y \in G$ say x is **conjugate** to y in G if

$$\exists g \in G : y = gxg^{-1}.$$

Show that conjugacy is an equivalence relation.
Its equivalence classes are called the **conjugacy classes** of G.

Solution. Let $x, y, z, g, h \in G$.

- reflexive: $1x1^{-1} = x$. So x is conjugate to itself.
- symmetric: If $y = gxg^{-1}$, then $g^{-1}yg = x$.
- transitive: If $y = gxg^{-1}$ and $z = hyh^{-1}$, then $z = hgx(gh)^{-1}$.

□

(2) (a) Show that a subgroup N of G is normal iff N is a union of conjugacy classes of G.
(b) Which conjugacy classes are contained in the center $Z(G)$?

Solution.

(a) Assume $N \trianglelefteq G$. Let $x \in N$. By normality every conjugate gxg^{-1} of x for $g \in G$ is in N. So N contains the whole conjugacy class of x. Hence N is the union of the conjugacy classes of its elements.
Conversely, assume $N \leq G$ is a union of conjugacy classes. Then for any $x \in N, g \in G$ we have $gxg^{-1} \in N$. Thus N is normal.

(b) If $x \in Z(G)$, then x is only conjugate to itself. So the conjugacy class of x is $\{x\}$. □

(3) Use (2) to determine all normal subgroups of S_3.
(a) What is $Z(S_3)$?
(b) Describe the quotient groups S_3/N for these normal subgroups up to isomorphism as simple as possible.

Solution. The conjugacy classes of S_3 are

$$\{()\} \quad \{(123), (132)\} \quad \{(12), (13), (23)\}.$$

Which unions of them give subgroups? Only

$$\{()\} \quad A_3 = \{(), (123), (132)\} \quad S_3.$$

(a) The center is the union of the 1-element classes, so $\{()\}$.
(b)

$$S_3/\{()\} \cong S_3 \quad S_3/A_3 \cong \mathbb{Z}_2 \quad S_3/S_3 \cong 1.$$

(4) Show that every subgroup H of index 2 in a group G is normal.
Hint: Look at the partition of G into cosets of H.

Solution. Let $g \in G \setminus H$. Since $|G : H| = 2$,

$$G = H \cup gH = H \cup Hg.$$

So $H = H$ and $gH = Hg$. Since left and right cosets of H in G are the same, H is normal. □

(5) Let N be a normal subgroup of G. Show
(a) If G is abelian, then G/N is abelian.
(b) If G is cyclic, then G/N is cyclic.
(c) Give a nonabelian group G with G/N and N abelian.

Solution.
(a) Assume G is abelian. For $x, y \in G$

$$xNyN = xyN = yxN = yNxN$$

So G/N is abelian
(b) Let $G = \langle a \rangle$. Then $G/N = \langle aN \rangle$.
(c) By (3b) S_3 is nonabelian but has a cyclic normal subgroup A_3 and quotient $S_3/A_3 \cong \mathbb{Z}_2$. □

(6) Determine the orders of the following quotient groups and whether they are cyclic.
(a) $\mathbb{Z}/\langle 10 \rangle$
(b) $\mathbb{Z}_{12}/\langle 6 \rangle$
(c) $\mathbb{Z} \times \mathbb{Z}/\langle (2,3) \rangle$
(d) $\mathbb{Z}_4 \times \mathbb{Z}_6 \,/\langle (2,2) \rangle$

Solution.
(a) $\mathbb{Z}/\langle 10 \rangle = \mathbb{Z}_{10}$ is cyclic of order 10.
(b) $\mathbb{Z}_{12}/\langle 6 \rangle \cong \mathbb{Z}_6$ is cyclic of order 6.
(c) $\mathbb{Z}\times\mathbb{Z}/\langle (2,3) \rangle$ is infinite and generated by the coset of $(1,1)$. Note that $\langle (1,1), (2,3) \rangle = \mathbb{Z}^2$ since $(1,0) = 3(1,1) - (2,3)$ and $(0,1) = -2(1,1) + (2,3)$.
(d) $\mathbb{Z}_4\times\mathbb{Z}_6 \,/\langle (2,2) \rangle$ has order 3 since $\langle (2,2) \rangle = \{(0,0), (2,2), (0,2), (2,4)\}$.
□

(7) Show that $\pi \colon G \times H \to G,\ (g,h) \mapsto g$ is a homomorphism.
Determine its kernel and image.
Show that $G \times H/\{1\} \times H \cong G$.

Solution. $\ker \pi = 1 \times H$ and the image is G. By the first isomorphism theorem $(G \times H)/(\{1\} \times H) \cong G$. □

(8) Determine the kernels and images of the following homomorphisms. Which are injective, surjective?
(a) $\varphi \colon \mathbb{Z}_6 \to \mathbb{Z}_6,\ x \mapsto 4x$

(b) $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2,\ \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$
for the additive group $\mathbb{R}^2$.

(c) $h\colon \mathbb{C}^* \to \mathbb{C}^*, x \mapsto x^2$, where $\mathbb{C}^*$ denotes the multiplicative group of the complex numbers without 0.

Solution.

(a) $\varphi(\mathbb{Z}_6) = 4\mathbb{Z}_6 = \{0, 4, 2\}$ and $\ker \varphi = \{0, 3\}$. Neither injective nor surjective.

(b) Recall Linear Algebra: The kernel of this linear map is the null space of the matrix, hence spanned by the vector $(3, 1)$. The image is the column space, hence spanned by the vector $(1, -2)$. Neither injective nor surjective.

(c) $\ker h = \{1, -1\}$ and h is not injective.
Recall from Calculus: Every complex number is a square (has a square root). So $h(\mathbb{C}^*) = \mathbb{C}^*$ and h is surjective.