Math 3140 - Assignment 7

Due March 6, 2024

(1) For a group G and $x, y \in G$ say x is **conjugate** to y in G if
$$\exists g \in G : y = gxg^{-1}.$$
Show that conjugacy is an equivalence relation.
Its equivalence classes are called the **conjugacy classes** of G.

(2) (a) Show that a subgroup N of G is normal iff N is a union of conjugacy classes of G.
(b) Which conjugacy classes are contained in the center $Z(G)$?

(3) Use (2) to determine all normal subgroups of S_3.
(a) What is $Z(S_3)$?
(b) Describe the quotient groups S_3/N for these normal subgroups up to isomorphism as simple as possible.

(4) Show that every subgroup H of index 2 in a group G is normal.
Hint: Look at the partition of G into cosets of H.

(5) Let N be a normal subgroup of G. Show
(a) If G is abelian, then G/N is abelian.
(b) If G is cyclic, then G/N is cyclic.
(c) Give a nonabelian group G with G/N and N abelian.

(6) Determine the orders of the following quotient groups and whether they are cyclic.
(a) $\mathbb{Z}/\langle 10\rangle$
(b) $\mathbb{Z}_{12}/\langle 6\rangle$
(c) $\mathbb{Z}\times\mathbb{Z}/\langle (2,3)\rangle$
(d) $\mathbb{Z}_4\times\mathbb{Z}_6\ /\langle (2,2)\rangle$

(7) Show that $\pi\colon G\times H \to G,\ (g,h)\to g$ is a homomorphism.
Determine its kernel and image.
Show that $G\times H/\{1\}\times H \cong G$.

(8) Determine the kernels and images of the following homomorphisms. Which are injective, surjective?
(a) $\varphi\colon \mathbb{Z}_6\to\mathbb{Z}_6,\ x\mapsto 4x$
(b) $\psi\colon \mathbb{R}^2\to\mathbb{R}^2,\ \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}\cdot\begin{bmatrix} x \\ y \end{bmatrix}$
for the additive group $\mathbb{R}^2$.
(c) $h\colon \mathbb{C}^*\to\mathbb{C}^*, x\mapsto x^2$, where $\mathbb{C}^*$ denotes the multiplicative group of the complex numbers without 0.