# Math 3140 - Assignment 6

Due February 28, 2024

(1) Let $G$ be a finite group with subgroups $H \leq K \leq G$. Show that
$$|G:H| = |G:K| \cdot |K:H|.$$
**Solution.** Recall that $|G:H| = \frac{|G|}{|H|}$, etc. So
$$|G:K| \cdot |K:H| = \frac{|G|}{|K|} \cdot \frac{|K|}{|H|} = \frac{|G|}{|H|} = |G:H|.$$
□

(2) For groups $(G, \cdot_G)$ and $(H, \cdot_H)$ show that the *direct product*
$$G \times H := \{(g,h) \ : \ g \in G, h \in H\}$$
under the operation
$$(g_1, h_1) \cdot (g_2, h_2) := (g_1 \cdot_G g_2, h_1 \cdot_H h_2)$$
for $g_1, g_2 \in G, h_1, h_2 \in H$ is a group.

**Solution.**
(a) Clearly $\cdot$ is a function from $(G \times H)^2$ to $G \times H$.
(b) $\cdot$ inherits associativity from the multiplications on $G$ and on $H$.
(c) $(1_G, 1_H)$ is the identity of $G \times H$ where $1_G, 1_H$ is the identity of $G, H$, respectively.
(d) Any $(g,h) \in G \times H$ has an inverse $(g,h)^{-1} = (g^{-1}, h^{-1})$.
□

(3) Give an isomorphism between the following pairs of groups or explain why they are not isomorphic.
(a) $\mathbb{Z}_8 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_4$
(b) $\mathbb{Z}_{12}$, $\mathbb{Z}_4 \times \mathbb{Z}_3$
(c) $\mathbb{Z}_{12}^*$, $\mathbb{Z}_4^* \times \mathbb{Z}_3^*$
(d) $Z_4 \times \mathbb{Z}_2$, $D_8$

**Solution.**
(a) $\mathbb{Z}_8 \times \mathbb{Z}_2 \not\cong \mathbb{Z}_4 \times \mathbb{Z}_4$ since $(1,0) \in \mathbb{Z}_8 \times \mathbb{Z}_2$ has order 8 but all elements of $\mathbb{Z}_4 \times \mathbb{Z}_4$ have order dividing 4
(b) $\mathbb{Z}_{12} \to \mathbb{Z}_4 \times \mathbb{Z}_3, \ [x]_{12} \mapsto ([x]_3, [x]_4)$ is an isomorphism.
(c) $\mathbb{Z}_{12}^* \to \mathbb{Z}_4^* \times \mathbb{Z}_3^*, \ [x]_{12} \mapsto ([x]_3, [x]_4)$ is an isomorphism.
(d) $Z_4 \times \mathbb{Z}_2 \not\cong D_8$ since the first is abelian, the second not

(4) Give an isomorphism between the following pairs of groups or explain why they are not isomorphic.
   (a) $G \times H$, $H \times G$
   (b) $\mathbb{C}$, $\mathbb{R} \times \mathbb{R}$ under addition
   (c) $\mathbb{C}^*$, $\mathbb{R}^* \times \mathbb{R}^*$ under multiplication
   (d) $\mathbb{Z} \times \mathbb{Z}$, $(\{2^x 3^y \ : \ x, y \in \mathbb{Z}\}, \cdot)$

   **Solution.**
   (a) $G \times H \times H \times G,\ (g, h) \mapsto (h, g)$, is an isomorphism.
   (b) $\mathbb{C} \to \mathbb{R} \times \mathbb{R},\ a + bi \mapsto (a, b)$, is an isomorphism since addition of complex numbers is componentwise.
   (c) $\mathbb{C}^* \ncong \mathbb{R}^* \times \mathbb{R}^*$ since e.g. $-1$ is the unique element of order 2 in $\mathbb{C}^*$ but $(-1, 1), (1, -1), (-1, -1)$ all have order 2 in $\mathbb{R}^* \times \mathbb{R}^*$
   (d) $\mathbb{Z} \times \mathbb{Z} \to (\{2^x 3^y \ : \ x, y \in \mathbb{Z}\}, \cdot),\ (x, y) \mapsto 2^x 3^y$, is an isomorphism.

(5) Note that $1 \times H$ is a subgroup of $G \times H$. Find all the left cosets of $1 \times H$ in $G \times H$. Give one representative for each left coset. How many are there?

   **Solution.** Every left coset of $1 \times H$ in $G \times H$ is of the form $\{g\} \times H$ for $g \in G$.

   So $(g, 1)$ for $g \in G$ is a set of representatives through all cosets. There are as many as elements in $G$. □

(6) Let $p$ be a prime. Show that $\mathbb{Z}_p \times \mathbb{Z}_p$ has exactly $p+1$ subgroups of order $p$.

   **Solution.** Every nonzero element in $\mathbb{Z}_p \times \mathbb{Z}_p$ has order $p$. So there are $p^2 - 1$ elements of order $p$, each of which generates a subgroup of order $p$. However any subgroup of order $p$ has $p - 1$ distinct generators. So the number of distinct subgroups of order $p$ is $\frac{p^2-1}{p-1} = p + 1$. □

(7) Let $p, q$ be odd primes and let $m, n \in \mathbb{N}$. Show that $\mathbb{Z}^*_{p^m} \times \mathbb{Z}^*_{q^n}$ is not cyclic.

   Hint: Note that a cyclic group has at most one element of order 2. Why?

   **Solution.** An infinite cyclic group has no element of order 2. By a Theorem from class $\mathbb{Z}_k$ has a subgroup $H$ of order 2 iff $k$ is even. Then $H = \langle \frac{k}{2} \rangle$, in particular, $\frac{k}{2}$ is the only element of order 2.

   For $p, q$ odd primes, note that $(-1, 1), (1, -1), (-1, -1)$ are 3 distinct elements in $\mathbb{Z}^*_{p^m} \times \mathbb{Z}^*_{q^n}$ of order 2. Hence that group cannot be cyclic. □