Math 3140 - Assignment 6

Due February 28, 2024

(1) Let G be a finite group with subgroups $H \leq K \leq G$. Show that
$$|G : H| = |G : K| \cdot |K : H|.$$

(2) For groups $(G, \cdot_G)$ and $(H, \cdot_H)$ show that the *direct product*
$$G \times H := \{(g, h) \ : \ g \in G, h \in H\}$$
under the operation
$$(g_1, h_1) \cdot (g_2, h_2) := (g_1 \cdot_G g_2, h_1 \cdot_H h_2)$$
for $g_1, g_2 \in G, h_1, h_2 \in H$ is a group.

(3) Give an isomorphism between the following pairs of groups or explain why they are not isomorphic.
  (a) $\mathbb{Z}_8 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_4$
  (b) $\mathbb{Z}_{12}$, $\mathbb{Z}_4 \times \mathbb{Z}_3$
  (c) $\mathbb{Z}_{12}^*$, $\mathbb{Z}_4^* \times \mathbb{Z}_3^*$
  (d) $Z_4 \times \mathbb{Z}_2$, D_8

(4) Give an isomorphism between the following pairs of groups or explain why they are not isomorphic.
  (a) $G \times H$, $H \times G$
  (b) $\mathbb{C}$, $\mathbb{R} \times \mathbb{R}$ under addition
  (c) $\mathbb{C}^*$, $\mathbb{R}^* \times \mathbb{R}^*$ under multiplication
  (d) $\mathbb{Z} \times \mathbb{Z}$, $(\{2^x 3^y \ : \ x, y \in \mathbb{Z}\}, \cdot)$

(5) Note that $1 \times H$ is a subgroup of $G \times H$. Find all the left cosets of $1 \times H$ in $G \times H$. Give one representative for each left coset. How many are there?

(6) Let p be a prime. Show that $\mathbb{Z}_p \times \mathbb{Z}_p$ has exactly $p+1$ subgroups of order p.

(7) Let p, q be odd primes and let $m, n \in \mathbb{N}$. Show that $\mathbb{Z}_{p^m}^* \times \mathbb{Z}_{q^n}^*$ is not cyclic.

Hint: Note that a cyclic group has at most one element of order 2. Why?