Math 3140 - Assignment 5

Due February 21, 2024

These problems are review for Midterm 1 on February 21. Do them before the exam!

(1) Compute the multiplicative inverses of the following if they exist:
   (a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ in $\mathrm{GL}(2, \mathbb{R})$
   (b) $b = (2\ 3\ 4)(1\ 2\ 3)$ in S_4 (give a decomposition in disjoint cycles)
   (c) $c = [9]$ in $\mathbb{Z}_{25}$

(2) Prove or disprove:
$\mathbb{Z}$ with the operation $x \oplus y := x + y + 3$ for $x, y \in \mathbb{Z}$ is a group.

(3) For G a permutation group on X and $x \in X$, the **stabilizer** of x in G is

$$\mathrm{stab}_G(x) := \{g \in G \ : \ g(x) = x\}.$$

Show that $\mathrm{stab}_G(x)$ is a subgroup of G.

(4) The **exponent** of a group G is the smallest $n > 0$ such that $g^n = 1$ for all $g \in G$ if it exists; else the exponent of G is infinite.
Show that every group G of exponent 2 is abelian.

(5) Let $(G, \cdot)$ be a group and $g \in G$. Show that

$$a_g \colon G \to G,\ x \mapsto gx,$$

is a permutation on G. Is a_g a homomorphism on G?

(6) Let $(G, \cdot)$ be a group and $S \subseteq G$. Show that the subgroup generated by S consists of all finite products of integer powers of elements in S, i.e.

$$\langle S \rangle = \{a_1^{k_1} \cdots a_n^{k_n} \ : \ n \geq 0, k_1, \ldots, k_n \in \mathbb{Z}, a_1, \ldots, a_n \in S\}$$

Hint: Show that if a subgroup H contains S, then it also contains all products of powers of elements in S.
Conversely, show that the set of products of powers of elements in S is a subgroup of G.

(7) Assume that a group G contains elements of all orders between 1 and 10. What is the smallest possible order of G?

(8) Let G be a nontrivial group that has no proper, nontrivial subgroups (i.e. 1 and G are the only subgroups of G). Show that $|G|$ is prime.

Hint: Do not assume at the outset that G is finite.