Math 3140 - Assignment 4

Due February 14, 2024

(1) Let $\mathbb{C}$ be the set of complex numbers and

$$M = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \ : \ a, b \in \mathbb{R} \right\}.$$

Show that $(\mathbb{C}, +) \cong (M, +)$ and $(\mathbb{C} \setminus \{0\}, \cdot) \cong (M \setminus \{0\}, \cdot)$.

(2) Let G be a group. Show that $\mathrm{Aut}G$ is group a under composition of functions.

(3) For a group G and $g \in G$, define the inner automorphism

$$\varphi_g \colon G \to G,\ x \mapsto gxg^{-1}.$$

Show

(a) $\varphi_g \in \mathrm{Aut}G$.

(b) $\Phi \colon G \to \mathrm{Aut}G,\ g \mapsto \varphi_g$, is a homomorphism.

(c) $\ker \Phi = Z(G)$.

(4) Let $D_8 = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$ be the dihedral group of order 8 generated by a rotation a and a reflection b.

(a) When are $\varphi_g = \varphi_h$ for $g, h \in D_8$?

(b) What is the order of $\mathrm{Inn}D_8$?

(c) List all distinct elements in $\mathrm{Inn}D_8$.

(5) Show that $|\mathrm{Aut}D_8| \leq 8$.

Hint: Explain why an automorphism is uniquely determined by what it does to the generators a and b. Where could these be mapped to?

(6) Find non-isomorphic groups G, H such that $\mathrm{Aut}G \cong \mathrm{Aut}H$.

(7) For the following subgroups H of G, find all the left cosets of H in G. Give one representative for each left coset. How many are there?

(a) $G = \mathbb{R}^2$ under addition, $H = \{(x, 0) \ : \ x \in \mathbb{R}\}$

(b) $G = \langle a \rangle$ of order 12, $H = \langle a^4 \rangle$

(c) $G = \mathbb{R}^*$ under multiplication, $H = \mathbb{R}^+$ the subgroup of positive reals

(8) For any integer $n > 1$, Euler's ϕ-function $\phi(n)$ yields the number of positive integers less than n that are coprime to n. Prove:

Euler's Theorem. If a is coprime to n, then $a^{\phi(n)} \equiv 1 \bmod n$.