Math 3140 - Assignment 3

Due February 7, 2024

(1) For permutations $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{pmatrix}$ compute

$$\alpha^{-1}, \alpha\beta, \beta\alpha$$

and their orders.

(2) Find all subgroups of S_3, give generators for each and show their inclusions in a subgroup lattice.

Hint: First find all cyclic subgroups. Then all subgroups that need two generators, etc.

Solution: trivial subgroup: 1

cyclic subgroups: $\langle(12)\rangle \quad \langle(13)\rangle \quad \langle(23)\rangle \quad \langle(123)\rangle$

any 2 distinct non-identity permutations generate S_3 □

(3) Show that a cycle of length ℓ can be written as a product of $\ell - 1$ transpositions,

$$(a_1\ a_2\ \ldots a_\ell) = (a_1\ a_\ell)(a_1 a_{\ell-1})\ldots(a_1 a_2).$$

Solution: Note that the functions on each side map

$a_1 \mapsto a_2$

$a_2 \mapsto a_3$

$\vdots$

$a_\ell \mapsto a_1$

and fix all other points. Hence they are equal. □

(4) Show that $Z(S_n) = 1$ for $n \geq 3$.

Solution: $\supseteq$: Clearly the identity () commutes with all elements in S_n, hence is in its center.

$\subseteq$: Let $\alpha \in S_n$ such that $\alpha \neq ()$. Show that there is some $\beta \in S_n$ such that $\alpha\beta \neq \beta\alpha$. For that β needs to move the same points as α but differently.

Since $\alpha \neq ()$, we have $x \in \{1, \ldots, n\}$ such that $\alpha(x) \neq x$. Let $y \in \{1, \ldots n\}$ be distinct from x and $\alpha(x)$. Such y exists since $n \geq 3$. Let $\beta := (x, y)$.

Then $\alpha\beta(x) = \alpha(y)$ and $\beta\alpha(x) = \alpha(x)$. Note $\alpha(x) \neq \alpha(y)$ since $x \neq y$. So $\alpha\beta \neq \beta\alpha$ and $\alpha \notin Z(S_n)$. Thus $Z(S_n) \subseteq 1$. □

(5) Let $\varphi\colon G \to H$ be a homomorphism between the group $(G, \cdot)$ with identity 1_G and the group $(H, *)$ with identity 1_H. Show:
 (a) $\varphi(1_G) = 1_H$.
 Hint: Start by evaluating $\varphi(1_G \cdot 1_G)$ in two ways.
 (b) $\varphi(g^{-1}) = \varphi(g)^{-1}$ for all $g \in G$.
 Hint: Use (a).

Solution: (a)

$$\varphi(1_G) = \varphi(1_G \cdot 1_G) = \varphi(1_G) * \varphi(1_G)$$

Multiply by $\varphi(1_G)^{-1}$ in H to get $1_H = \varphi(1_G)$.

(b) $\varphi(g^{-1}) \cdot \varphi(g) = \varphi(g^{-1}g) = \varphi(1_G) = 1_H$ by (a) yields that $\varphi(g^{-1})$ is the inverse of $\varphi(g)$. □

(6) Let G, H be groups and $\varphi\colon G \to H$ a homomorphism. Show
 (a) The image of φ,

$$\varphi(G) := \{\varphi(g) \ : \ g \in G\}$$

 is a subgroup of H.
 (b) The kernel of φ,

$$\ker\varphi := \{g \in G \ : \ \varphi(g) = 1\}$$

 is a subgroup of G.

Recall image and kernel of linear maps on vector spaces.

Solution: (a) Show that $\varphi(G)$ is closed under multiplication and inverses:

Let $a, b \in \varphi(G)$. Then we have $x, y \in G$ such that $a = \varphi(x), b = \varphi(y)$. Now

$$ab = \varphi(x)\varphi(y) = \varphi(xy) \in \varphi(G),$$

$$a^{-1} = \varphi(x^{-1}) \in \varphi(G) \text{ by (5a).}$$

(b) Show that $\ker\varphi$ is closed under multiplication and inverses:

Let $x, y \in \ker\varphi$. Then $\varphi(x) = \varphi(y) = 1_H$ and

$$\varphi(xy) = \varphi(x)\varphi(y) = 1_H \text{ yields } xy \in \ker\varphi$$

$$x^{-1} \in \ker\varphi \text{ since by (5b) } \varphi(x^{-1}) = \varphi(x)^{-1} = 1_H.$$

□

(7) Let G be a group. Show that

$$\varphi\colon G \to G,\ x \mapsto x^{-1},$$

is an automorphism iff G is abelian.

Solution: Note that φ is its own inverse since $\varphi(\varphi(x)) = x$. Hence φ is always bijective.

First assume G is commutative. Let $x, y \in G$. Then

$$\varphi(xy) = (xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1} = \varphi(x)\varphi(y),$$

shows that φ is a homomorphism.

Conversely assume φ is a homomorphism. Then $\varphi(x)\varphi(y) = \varphi(xy)$, which means

$$x^{-1}y^{-1} = (xy)^{-1} = y^{-1}x^{-1}.$$

Multiplying with x, y in the correct order yields $xy = yx$ and G is commutative. □

(8) Are the groups G and H isomorphic? If no, explain why not. If yes, give an explicit isomorphism $\varphi\colon G \to H$:

(a) $G = (\mathbb{Z}, +)$, $H = (2\mathbb{Z}, +)$
(b) $G = (\mathbb{Z}_6, +)$, $H = (\mathbb{Z}_7^*, \cdot)$.
(c) G the symmetry group of a rectangle, $H = (\mathbb{Z}_4, +)$
(d) $G = S_3$, $H = \mathbb{Z}_6$

Solution:

(a) $\mathbb{Z}$ is infinite cyclic with generator 1, $2\mathbb{Z}$ is infinite cyclic with generator 2. So $\varphi\colon G \to H,\ x \mapsto 2x$, is an isomorphism.
(b) $G = (\mathbb{Z}_6, +)$ is cyclic of order 6 with generator $[1]_6$ and $H = (\mathbb{Z}_7^*, \cdot)$ is cyclic of order 6 with generator $[3]_7$. So $\varphi\colon G \to H,\ [x]_6 \mapsto [3]_7^x$, is an isomorphism.
(c) Recall that G contains 3 flips of the rectangle of order 2. Hence G is not cyclic and cannot be isomorphic to $\mathbb{Z}_4$.
(d) S_3 is non-abelian and cannot be isomorphic to the abelian group $\mathbb{Z}_6$. □