Math 3140 - Assignment 3

Due February 7, 2024

(1) For permutations $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{pmatrix}$ compute

$$\alpha^{-1}, \alpha\beta, \beta\alpha$$

and their orders.

(2) Find all subgroups of S_3, give generators for each and show their inclusions in a subgroup lattice.

Hint: First find all cyclic subgroups. Then all subgroups that need two generators, etc.

(3) Show that a cycle of length ℓ can be written as a product of $\ell - 1$ transpositions,

$$(a_1\ a_2\ \dots a_\ell) = (a_1\ a_\ell)(a_1 a_{\ell-1}) \dots (a_1 a_2).$$

(4) Show that $Z(S_n) = 1$ for $n \geq 3$.

(5) Let $\varphi\colon G \to H$ be a homomorphism between the group $(G, \cdot)$ with identity 1_G and the group $(H, *)$ with identity 1_H. Show:

(a) $\varphi(1_G) = 1_H$.
Hint: Start by evaluating $\varphi(1_G \cdot 1_G)$ in two ways.

(b) $\varphi(g^{-1}) = \varphi(g)^{-1}$ for all $g \in G$.
Hint: Use (a).

(6) Let G, H be groups and $\varphi\colon G \to H$ a homomorphism. Show

(a) The image of φ,

$$\varphi(G) := \{\varphi(g)\ :\ g \in G\}$$

is a subgroup of H.

(b) The kernel of φ,

$$\ker \varphi := \{g \in G\ :\ \varphi(g) = 1\}$$

is a subgroup of G.

Recall image and kernel of linear maps on vector spaces.

(7) Let G be a group. Show that

$$\varphi\colon G \to G,\ x \mapsto x^{-1},$$

is an automorphism iff G is abelian.

(8) Are the groups G and H isomorphic? If no, explain why not. If yes, give an explicit isomorphism $\varphi\colon G \to H$:

(a) $G = (\mathbb{Z}, +)$, $H = (2\mathbb{Z}, +)$

(b) $G = (\mathbb{Z}_6, +)$, $H = (\mathbb{Z}_7^*, \cdot)$.

(c) G the symmetry group of a rectangle, $H = (\mathbb{Z}_4, +)$

(d) $G = S_3$, $H = \mathbb{Z}_6$