Math 3140 - Assignment 2

Due January 31, 2024

(1) Use the Euclidean algorithm to find $\gcd(a, b)$ and Bezout's coefficients $u, v \in \mathbb{Z}$ such that

$$u \cdot a + v \cdot b = \gcd(a, b)$$

for $a = 51, b = 36$.

(2) Compute the following multiplicative inverses in $\mathbb{Z}_n$ if possible:
 (a) $[12]^{-1}$ in $\mathbb{Z}_{35}$
 (b) $[14]^{-1}$ in $\mathbb{Z}_{35}$
 Hint: Use the Euclidean Algorithm to compute Bezout's coefficients.

(3) For $n \in \mathbb{N}$, let $\mathbb{Z}_n^*$ denote the set of elements in $\mathbb{Z}_n$ that have a multiplicative inverse. Show that $(\mathbb{Z}_n^*, \cdot)$ is a group.
 Hint: Don't forget to show that $\cdot$ is an operation on $\mathbb{Z}_n^*$, i.e., that the product of invertible elements is invertible again.

(4) Let A, B be subgroups of a group $(G, \cdot)$. Show that $A \cap B$ is a subgroup as well.

(5) Determine the center of $\mathrm{GL}(2, \mathbb{R})$.
 Hint: Let E_{ij} be the 2×2 matrix whose ij-entry is 1 and all other entries are 0. This is not invertible but their sum with the identity matrix $I + E_{ij}$ is.
 Note that $A(I + E_{ij}) = (I + E_{ij})A$ iff $AE_{ij} = E_{ij}A$. Check the latter equations to determine conditions on a, b, c, d such that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Z(\mathrm{GL}(2, \mathbb{R}))$.

(6) Prove that every group of even order has an element of order 2.

(7) Which of the following groups are cyclic? For those that are, list all their generators. For those that are not, explain why.
 $A = (\mathbb{Q}, +)$
 $B = (\mathbb{Z}_{12}, +)$
 $C = (\mathbb{Z}_7^*, \cdot)$
 $D = \{\pi^z \ : \ z \in \mathbb{Z}\}$ under multiplication
 $E = \mathbb{Z}^2 = \{(a, b) \ : \ a, b \in \mathbb{Z}\}$ under addition

(8) How many subgroups does $(\mathbb{Z}_{20}, +)$ have? List a generator for each subgroup. Draw a diagram showing the containments between the subgroups.