Math 3140 - Assignment 11

Due November 13, 2026

(1) Which of the following are group actions? Check the properties. Are they transitive?
   (a) G on $X := G/H$ for a subgroup H of G by $g * xH := gxH$
   (b) G on $X := G$ by $g * x := g^{-1}xg$

Solution.
   (a) Group action because 1 fixes all cosets xG and $(gh)xH = g(hxH)$ for all $g, h, x \in G$.
   There is just one orbit since for any $g \in G$, the coset H can be translated into gH. Hence the action is transitive.
   (b) In general not a group action since $(gh) * x = (gh)^{-1}xgh$ whereas $g * (h * x) = g^{-1}h^1xhg$.
   Note that conjugation becomes a group action when defining it by $g * x := gxg^{-1}$. □

(2) Which of the following are group actions of $\mathrm{GL}(n, \mathbb{R})$? Check the properties. What are the orbits?
   (a) $A * x := A \cdot x$ for $A \in \mathrm{GL}(n, \mathbb{R}), x \in R^n$,
   (b) $A * (b_1, \ldots, b_n) := (Ab_1, \ldots, Ab_n)$ for $A \in \mathrm{GL}(n, \mathbb{R})$ and a basis $(b_1, \ldots, b_n)$ of $\mathbb{R}^n$.

Solution.
   (a) Group action because the identity matrix I_n fixes all $x \in \mathbb{R}^n$ and $(AB)x = A(Bx)$ for all $A, B \in \mathrm{GL}(n, \mathbb{R}^n), x \in \mathbb{R}^n$.
   The zero vector 0 is in its own orbit. Else, the orbit of the first unit vector $e_1 := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is $\mathbb{R}^n \setminus \{0\}$ because Ae_1 is just the first column of A, which can be any non-zero vector. Hence there are precisely 2 orbits in total.
   (b) Group action because the identity matrix I_n fixes every basis and $(AB)x = A(Bx)$ for all $A, B \in \mathrm{GL}(n, \mathbb{R}^n), x \in \mathbb{R}^n$.
   For the standard basis with unit vectors $(e_1, \ldots, e_n)$ we see that $A(e_1, \ldots, e_n) = (Ae_1, \ldots, Ae_n)$, which is just the tuple of columns of A. Recall that a matrix A is invertible iff its columns form a basis. So the orbit of the standard basis is the set of all basis. This action is transitive. □

(3) For $(G, \cdot)$ acting on a set X and $x, y \in X$, define $x \sim y$ if $\exists g \in G\colon y = gx$. Show:
   (a) $\sim$ is an equivalence relation on X.
   (b) The orbit $Gx := \{gx : g \in G\}$ is the equivalence class of x with respect to $\sim$.

Solution.
   (a) Reflexive because $x = 1x$ for all $x \in X$.
   Symmetric because $y = gx$ implies $x = g^{-1}y$.
   Transitive because $y = gx$ and $z = hy$ imply $z = hgx$.
   (b) follows since $\{y : y = gx \text{ for some } g \in G\} = \{gx : g \in G\}$. □

(4) (a) How many distinct necklaces can be made with 2 red, 2 blue and 2 green beads?
   (b) How many distinct necklaces can be made with 6 beads of (at most) 3 different colors?

Solution: For (a) we have D_{12} act on the set X of all possible arrangements of 2 red, 2 blue and 2 green beads, that is $|X| = \binom{6}{2}\binom{4}{2}\binom{2}{2} = 90$.

For (b) we have D_{12} act on the set Y of all possible colorings of 6 vertices in red, blue and green, that is $|Y| = 3^6$.

We list how often each cycle structure occurs in D_{12} and how many fixed points the corresponding permutation has in X, Y, respectively. The first 4 rows correspond to rotations, the last 2 to reflections of the hexagon. Note that in every cycle the fixed elements have to have the same color.

multiplicity	$g \in D_{12}$	$\lvert\mathrm{Fix}(g)\rvert$ in X	$\lvert\mathrm{Fix}(g)\rvert$ in Y
1	()	90	3^6
2	(1 2 3 4 5 6)	0	3^1
2	(1 3 5)(2 4 6)	0	3^2
1	(1 4)(2 5)(3 6)	3!	3^3
3	(1 2)(3 6)(2 5)	3!	3^3
3	(2 6)(3 5)(1)(4)	3!	3^4

By the Burnside-Frobenius Lemma we get

$$\frac{1}{12}(90 + 2 \cdot 0 + 2 \cdot 0 + 1 \cdot 6 + 3 \cdot 6 + 3 \cdot 6) = 11$$

orbits for (a). Similarly 92 orbits for (b). □

(5) Recall that the rotation group of a regular tetrahedron acts on the 4 vertices (equivalently the 4 faces) like A_4.

(a) In how many ways can the faces of a regular tetrahedron be colored with 4 colors so that every color occurs exactly once?
Try to draw the colorings and explain your result geometrically.
(b) In how many ways can the faces of a regular tetrahedron be colored with 4 colors without any restrictions?

Solution: For (a) we have A_4 act on the set X of all colorings X in 4 distinct colors, that is $|X| = 4!$.

For (b) we have A_4 act on the set Y of all possible colorings in 4 colors, that is $|Y| = 4^4$.

We list how often each cycle structure occurs in A_4 and how many fixed points the corresponding permutation has in X, Y, respectively. In every cycle the fixed elements have to have the same color.

multiplicity	$g \in A_4$	$\lvert\mathrm{Fix}(g)\rvert$ in X	$\lvert\mathrm{Fix}(g)\rvert$ in Y
1	()	$4!$	4^4
3	(1 2)(3 4)	0	4^2
8	(1 2 3)	0	4^2

By the Burnside-Frobenius Lemma we get

$$\frac{1}{12}(24 + 2 \cdot 0 + 3 \cdot 0 + 8 \cdot 0) = 2$$

orbits for (a).

For (a) note that coloring the faces numbered $1, 2, 3, 4$ of the tetrahedron in colors called $1, 2, 3, 4$ can be considered as a permutation of $1, 2, 3, 4$. The action of A_4 maps "even" colorings to "even" colorings as well as "odd" colorings to "odd" colorings. But it cannot change an "even" coloring to an "odd" one.

Geometrically A_4 cannot translate a coloring of the tetrahedron to the coloring of the mirrored tetrahedron. So there are exactly 2 orbits in (a).

For (b) we get

$$\frac{1}{12}(4^4 + 11 \cdot 4^2) = 36$$

orbits. □

(6) (a) How many colorings are there of the faces of a cube in 2 colors up to rotational symmetry? (Two colorings are considered equivalent when one can be obtained from the other by rotating the cube.)

(b) How many ways can you label the faces of a die $1, \ldots, 6$ up to rotational symmetry?

Hint: recall the rotation group of the cube from [1, Thm 7.4].

Solution. There are 24 rotational symmetries of the cube in total:

	g	multiplicity	for (a): Fix(g)	for (b): Fix
identity	()	1	2^6	6!
$90°, 270°$ around center of face	(1 2 3 4)	6	2^3	0
$180°$ around center of face	(1 3)(2 4)	3	2^4	0
$120°, 240°$ around diagonal	(1 2 6)(3 4 5)	8	2^2	0
$180°$ around center of sides	(1 4)(2 3)(5 6)	6	2^3	0

(a) We get $\frac{1}{24}(64 + 6 * 8 + 3 * 16 + 8 * 4 + 6 * 8) = 10$ colorings in 2 colors up to rotational symmetry.

(b) We get $\frac{720}{24} = 30$ labellings of the faces of a die up to rotational symmetry.

□

References

[1] Joseph A. Gallian. Contemporary Abstract Algebra. Houghton Mifflin Company, sixth edition, 2006.