Math 3140 - Assignment 11

Due November 13, 2026

(1) Which of the following are group actions? Check the properties. Are they transitive?
   (a) G on $X := G/H$ for a subgroup H of G by $g * xH := gxH$
   (b) G on $X := G$ by $g * x := g^{-1}xg$

(2) Which of the following are group actions of $\mathrm{GL}(n, \mathbb{R})$? Check the properties. What are the orbits?
   (a) $A * x := A \cdot x$ for $A \in \mathrm{GL}(n, \mathbb{R}), x \in R^n$,
   (b) $A * (b_1, \ldots, b_n) := (Ab_1, \ldots, Ab_n)$ for $A \in \mathrm{GL}(n, \mathbb{R})$ and a basis $(b_1, \ldots, b_n)$ of $\mathbb{R}^n$.

(3) For $(G, \cdot)$ acting on a set X and $x, y \in X$, define $x \sim y$ if $\exists g \in G \colon y = gx$. Show:
   (a) $\sim$ is an equivalence relation on X.
   (b) The orbit $Gx := \{gx : \ g \in G\}$ is the equivalence class of x with respect to $\sim$.

(4) (a) How many distinct necklaces can be made with 2 red, 2 blue and 2 green beads?
   (b) How many distinct necklaces can be made with 6 beads of (at most) 3 different colors?

(5) Recall that the rotation group of a regular tetrahedron acts on the 4 vertices (equivalently the 4 faces) like A_4.
   (a) In how many ways can the faces of a regular tetrahedron be colored with 4 colors so that every color occurs exactly once?
   Try to draw the colorings and explain your result geometrically.
   (b) In how many ways can the faces of a regular tetrahedron be colored with 4 colors without any restrictions?

(6) (a) How many colorings are there of the faces of a cube in 2 colors up to rotational symmetry? (Two colorings are considered equivalent when one can be obtained from the other by rotating the cube.)
   (b) How many ways can you label the faces of a die $1, \ldots, 6$ up to rotational symmetry?

   Hint: recall the rotation group of the cube from [1, Thm 7.4].

□

References

[1] Joseph A. Gallian. Contemporary Abstract Algebra. Houghton Mifflin Company, sixth edition, 2006.