Math 3140 - Assignment 10

Due October 30, 2026

(1) For a prime p prime, find all abelian groups of order p^5 up to isomorphism.

Solution.

partitions of 5	groups
5	$\mathbb{Z}_{p^5}$
$4+1$	$\mathbb{Z}_{p^4} \times \mathbb{Z}_p$
$3+2$	$\mathbb{Z}_{p^3} \times \mathbb{Z}_{p^2}$
$3+1+1$	$\mathbb{Z}_{p^3} \times \mathbb{Z}_p \times \mathbb{Z}_p$
$2+2+1$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2} \times \mathbb{Z}_p$
$2+1+1+1$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$
$1+1+1+1+1$	$\mathbb{Z}_p^5$

(2) (a) Find all abelian groups of order 360 up to isomorphism.
(b) Which of these groups have exactly 3 elements of order 2?

Solution. (a) $360 = 2^3 \cdot 3^2 \cdot 5$. Partitioning the exponents of the prime powers

$$3 = 2+1 = 1+1+1, \quad 2 = 1+1, \quad 1,$$

we see that we have 3 partitions of 3, 2 partitions of 2 and 1 partition of 1. The $3 * 2 * 1$ distinct combinations of these partitions yields 6 non-isomorphic abelian groups of order 360:

$\mathbb{Z}_{2^3} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5$	$\mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5$	$(\mathbb{Z}_2)^3 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5$
$\mathbb{Z}_{2^3} \times (\mathbb{Z}_3)^2 \times \mathbb{Z}_5$	$\mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times (\mathbb{Z}_3)^2 \times \mathbb{Z}_5$	$(\mathbb{Z}_2)^3 \times (\mathbb{Z}_3)^2 \times \mathbb{Z}_5$

(b) Since $\mathbb{Z}_{2^3}$ has exactly 1 element of order 2, so do the groups in the first column above.

Since $\mathbb{Z}_{2^2} \times \mathbb{Z}_2$ has exactly 3 elements of order 2 (namely $(0,1), (2,0), (2,1)$), so do the groups in the middle column.

Since $\mathbb{Z}_2^3$ has exactly 7 elements of order 2 (all but $(0,0,0)$), so do the groups in the last column.

(3) Are the following groups isomorphic?
(a) $\mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
(b) $\mathbb{Z}_{10} \times \mathbb{Z}_{12} \times \mathbb{Z}_6$ and $\mathbb{Z}_{60} \times \mathbb{Z}_6 \times \mathbb{Z}_2$
(c) $\mathbb{Z}_{10} \times \mathbb{Z}_{12} \times \mathbb{Z}_6$ and $\mathbb{Z}_{15} \times \mathbb{Z}_4 \times \mathbb{Z}_{12}$

Solution (a) No, since the first has 3 elements of order 2 and the second 7 (or by the Fundamental Theorem of Finite Abelian Groups).
(b) Yes, splitting into prime power factors shows that both are isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$.

(c) No. The first splits into factors $\mathbb{Z}_4 \times \mathbb{Z}_2^2 \times \mathbb{Z}_3^2 \times \mathbb{Z}_5$, the second into $\mathbb{Z}_4^2 \times \mathbb{Z}_3^2 \times \mathbb{Z}_5$. By the Fundamental Theorem of Finite Abelian Groups these groups are not isomorphic.

(4) How many abelian groups up to isomorphism are there of order
(a) 6,
(b) 15,
(c) 30,
(d) pq for distinct primes p, q
(e) n where n is a product of pairwise distinct primes?

Solution. The answer is always 1.

A product of pairwise distinct primes means $n = p_1 p_2 \dots p_k$ for distinct primes $p_1, \dots, p_k$. By the Fundamental Theorem of Finitely Generated Abelian Groups every abelian group of that order is isomorphic to

$$\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \dots \mathbb{Z}_{p_k},$$

which is isomorphic to $\mathbb{Z}_n$ since these factors all have coprime orders.

(5) Let G, H be groups. Show that

$$Z(G \times H) = Z(G) \times Z(H).$$

Solution. We need to show that a pair is in the center of a direct product iff its components are in the center of the corresponding factors. So let $g \in G, h \in H$. Then

$(g,h) \in Z(G \times H)$
$\Leftrightarrow \forall x \in G, y \in H : \ (g,h)(x,y) = (x,y)(g,h)$
$\Leftrightarrow \forall x \in G, y \in H : \ gx = xg, hy = yh$
$\Leftrightarrow g \in Z(G), h \in Z(H)$
$\Leftrightarrow (g,h) \in Z(G) \times Z(H)$ □

(6) (a) Describe all isomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_4 \times \mathbb{Z}_3$. How many are there?
(b) Show that every homomorphism from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$ is of the form $(x,y) \mapsto ax + by$ for some integers a, b.

Solution. (a) Every isomorphism has to map the generator 1 of $\mathbb{Z}_{12}$ to an element of order 12. So there are 4 isomorphisms determined by
$1 \mapsto (1,1)$
$1 \mapsto (1,2)$
$1 \mapsto (3,1)$
$1 \mapsto (3,2)$
(b) Clearly every map $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, $(x,y) \mapsto ax + by$, is a

homomorphism. It remains to show that every homomorphism φ is of this form. Assume $\varphi(1,0) = a$ and $\varphi(0,1) = b$. Let $(x,y) \in \mathbb{Z}^2$. Then

$$\varphi(x,y) = \varphi((x,0)+(0,y)) = \varphi(x,0) + \varphi(0,y)$$
$$\varphi(x(1,0)) + \varphi(y(0,1)) = x\varphi(1,0) + y\varphi(0,1) = ax + by$$

□

(7) Let N be a normal subgroup of G such that G/N is abelian. Show that G/N is abelian iff $x^{-1}y^{-1}xy \in N$ for all $x, y \in G$.

The expression $x^{-1}y^{-1}xy$ is called the *commutator* of x and y and denoted by $[x,y]$.

Solution. Let $x, y \in G$. Then

$$xNyN = yNxN \Leftrightarrow xyN = yxN \Leftrightarrow x^{-1}yxy \in N.$$

□

(8) Let G' be the subgroup of G that is generated by the set of all commutators $\{[x,y] : x,y \in G\}$. Then G' is called the *commutator subgroup* or *derived subgroup* of G. Show that G' is normal in G.

Hint: Show that any conjugate of a commutator is a commutator.

Solution. Let $x, y, g \in G$. Then

$$g[x,y]g^{-1} = gx^{-1}y^{-1}xyg^{-1} = gx^{-1}g^{-1}gy^{-1}g^{-1}gxg^{-1}gyg^{-1} = [gxg^{-1}, gyg^{-1}].$$

Hence every conjugate of a generator of G' is in G' again. So every conjugate of any element of G' is contained in G'. Thus G' is normal in G. □

(9) By (7) and (8): **The commutator subgroup G' is the unique smallest normal subgroup N of G such that G/N is abelian.**

Use this and what you know about normal subgroups of the following groups to determine G' for
(a) G abelian, (b) S_3, (c) D_8 (d) A_4.
You do not need to compute any commutators $[x,y]$ for this.

Solution. (a) G is abelian iff $G' = 1$.

(b) S_3/A_3 is abelian. The only normal subgroup smaller than A_3 is the trivial subgroup 1 and $S_3/1 \cong S_3$ is not abelian. So $S_3' = A_3$.

(c) Recall $D_8 = \langle a, b : a^4 = 1, b^2 = 1, bab^{-1} = a^{-1}$. $D_8/\langle a^2\rangle$ has size 4 and is abelian. The only normal subgroup smaller

than $\langle a^2 \rangle$ is the trivial subgroup 1 and $D_8/1 \cong D_8$ is not abelian. So $D_8' = \langle a^2 \rangle$.

(d) $A_4/V_4 \cong \mathbb{Z}_3$ is abelian and this is the largest abelian quotient of A_4. So $A_4' = V_4$. $\square$