# Math 3140 - Assignment 10

Due October 30, 2026

This assignment is a set of practice problems for the midterm exam on November 2.

(1) For a prime $p$ prime, find all abelian groups of order $p^5$ up to isomorphism.

(2) (a) Find all abelian groups of order 360 up to isomorphism.
(b) Which of these groups have exactly 3 elements of order 2?

(3) Are the following groups isomorphic?
(a) $\mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
(b) $\mathbb{Z}_{10} \times \mathbb{Z}_{12} \times \mathbb{Z}_6$ and $\mathbb{Z}_{60} \times \mathbb{Z}_6 \times \mathbb{Z}_2$
(c) $\mathbb{Z}_{10} \times \mathbb{Z}_{12} \times \mathbb{Z}_6$ and $\mathbb{Z}_{15} \times \mathbb{Z}_4 \times \mathbb{Z}_{12}$

(4) How many abelian groups up to isomorphism are there of order
(a) 6, (b) 15, (c) 30, (d) $pq$ for distinct primes $p, q$
(e) $n$ where $n$ is a product of pairwise distinct primes?

(5) Let $G, H$ be groups. Show that

$$Z(G \times H) = Z(G) \times Z(H).$$

(6) (a) Describe all isomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_4 \times \mathbb{Z}_3$. How many are there?
(b) Show that every homomorphism from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$ is of the form $(x, y) \mapsto ax + by$ for some integers $a, b$.

(7) Let $N$ be a normal subgroup of $G$ such that $G/N$ is abelian. Show that $G/N$ is abelian iff $x^{-1}y^{-1}xy \in N$ for all $x, y \in G$.

The expression $x^{-1}y^{-1}xy$ is called the *commutator* of $x$ and $y$ and denoted by $[x, y]$.

(8) Let $G'$ be the subgroup of $G$ that is generated by the set of all commutators $\{[x, y] : x, y \in G\}$. Then $G'$ is called the *commutator subgroup* or *derived subgroup* of $G$. Show that $G'$ is normal in $G$.

Hint: Show that any conjugate of a commutator is a commutator.

(9) By (7) and (8): **The commutator subgroup $G'$ is the unique smallest normal subgroup $N$ of $G$ such that $G/N$ is abelian.**

Use this and what you know about normal subgroups of the following groups to determine $G'$ for
(a) $G$ abelian, (b) $S_3$, (c) $D_8$ (d) $A_4$.
You do not need to compute any commutators $[x, y]$ for this.