Math 3140 - Assignment 9

Due October 23, 2026

(1) Determine the kernels and images of the following homomorphisms. Which are injective, surjective?
   (a) $\varphi\colon \mathbb{Z}_6 \to \mathbb{Z}_6,\ x \mapsto 4x$
   (b) $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2,\ \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$
   for the additive group $\mathbb{R}^2$.
   (c) $h\colon \mathbb{C}^* \to \mathbb{C}^*, x \mapsto x^2$, where $\mathbb{C}^*$ denotes the multiplicative group of the complex numbers without 0.

Solution.
   (a) $\varphi(\mathbb{Z}_6) = 4\mathbb{Z}_6 = \{0, 4, 2\}$ and $\ker\varphi = \{0, 3\}$. Neither injective nor surjective.
   (b) Recall Linear Algebra: The kernel of this linear map is the null space of the matrix, hence spanned by the vector $(3, 1)$. The image is the column space, hence spanned by the vector $(1, -2)$. Neither injective nor surjective.
   (c) $\ker h = \{1, -1\}$ and h is not injective.
   Recall from Calculus: Every complex number is a square (has a square root). So $h(\mathbb{C}^*) = \mathbb{C}^*$ and h is surjective. □

(2) Let $n \in \mathbb{N}$. Show that $\operatorname{sign}\colon S_n \to (\{-1, 1\}, \cdot)$ defined by

$$\operatorname{sign}(f) := \frac{\prod_{1\le i<j\le n}(f(j) - f(i))}{\prod_{1\le i<j\le n}(j - i)}$$

is a homomorphism.
Hint: Use that for $f, g \in S_n$ and $1 \le i < j \le n$

$$\frac{fg(j) - fg(i)}{g(j) - g(i)} = \frac{fg(i) - fg(j)}{g(i) - g(j)}.$$

Hence you can reorder these factors such that the denominator is always positive.

Solution: Since $f \in S_n$ just permutates the elements in $\{1, \dots, n\}$, $\prod_{1\le i<j\le n}(j - i)$ and $\prod_{1\le i<j\le n}(f(j) - f(i))$ have the same absolute value. Hence $\operatorname{sign} f$ is either 1 or -1.

To show that sign is a homomorphism, let $f, g \in S_n$ and consider

$$\begin{aligned}\operatorname{sign}(fg) &= \prod_{1\le i<j\le n} \frac{fg(j)-fg(i)}{j-i} \\ &= \prod_{1\le i<j\le n} \left(\frac{fg(j)-fg(i)}{g(j)-g(i)} \cdot \frac{g(j)-g(i)}{j-i}\right) \\ &= \prod_{1\le i<j\le n} \frac{fg(j)-fg(i)}{g(j)-g(i)} \cdot \underbrace{\prod_{1\le i<j\le n} \frac{g(j)-g(i)}{j-i}}_{\operatorname{sign}(g)}\end{aligned}$$

To see that the first factor in the product above is $\operatorname{sign} f$, note that

$$\frac{fg(j)-fg(i)}{g(j)-g(i)} = \frac{fg(i)-fg(j)}{g(i)-g(j)}.$$

Hence we can reorder each factor in $\prod_{1\le i<j\le n} \frac{fg(j)-fg(i)}{g(j)-g(i)}$ such that the denominator is positive. Doing that and using that g just permutes the elements in $\{1,\dots,n\}$ we obtain

$$\prod_{1\le i<j\le n} \frac{fg(j)-fg(i)}{g(j)-g(i)} = \prod_{1\le k<l\le n} \frac{f(l)-f(k)}{l-k}.$$

Thus $\operatorname{sign}(fg) = \operatorname{sign}(f)\operatorname{sign}(g)$ follows. □

(3) Show $\operatorname{sign}(f) = -1$ for any transposition $f = (a\ b)$ in S_n.

Hint: Count the number of inversions of f, that is, the pairs $1 \le x < y \le n$ such that $f(x) > f(y)$. Recall from class that $\operatorname{sign}(f) = (-1)^{\text{number of inversions of } f}$.

Solution. Let $f = (k, l)$ be a transposition with $1 \le k < l \le n$. To compute $\operatorname{sign} f$ we just need to count how many of the factors $f(j) - f(i)$ for $1 \le i < j \le n$ in the enumerator of the formula for $\operatorname{sign} f$ are negative.

Note that f fixes all pairs $i < j$ if both i, j are distinct from k, l. Consequently $f(j) - f(i)$ is negative only if one of i or j are k or l, in particular if

- $i = k$ and $j \in \{k+1, \dots, l-1\}$,
- $j = l$ and $i \in \{k+1, \dots, l-1\}$, or
- $i = k, j = l$.

So in total there are $2(l-k-1)+1$ instances where $f(j)-f(i)$ is negative. Since this number is odd, $\operatorname{sign} f = -1$.

Alternatively: it is easy to see that $\operatorname{sign}(1,2) = -1$. One can show that every transposition f in S_n is conjugate to $(1,2)$, that is, $f = g(1,2)g^{-1}$ for some $g \in S_n$. Since sign is a homomorphism into a commutative group, this implies

$$\begin{aligned}\operatorname{sign} f &= \operatorname{sign}(g(1,2)g^{-1}) = \operatorname{sign} g \operatorname{sign}(1,2) (\operatorname{sign} g)^{-1} \\ &= \operatorname{sign} g (\operatorname{sign} g)^{-1} \operatorname{sign}(1,2) = \operatorname{sign}(1,2) = -1.\end{aligned}$$

□

(4) When are two elements of S_n conjugate?

(a) Show that for any k-cycle $(a_1, a_2, \ldots, a_k) \in S_n$ and any $f \in S_n$, we have

$$f(a_1, a_2, \ldots, a_k)f^{-1} = (f(a_1), f(a_2), \ldots, f(a_k)).$$

(b) For any two k-cycles $(a_1, a_2, \ldots, a_k), (b_1, b_2, \ldots, b_k) \in S_n$ explicitly give $f \in S_n$, such that

$$f(a_1, a_2, \ldots, a_k)f^{-1} = (b_1, b_2, \ldots, b_k).$$

The *cycle structure* of a permutation g is the length of the cycles in the cycle decomposition of g (counted with multiplicity). For example $g = (1\ 2\ 3)(4\ 5)(6\ 7)$ has cycle structure $3, 2, 2$.

Deduce that two permutations $g, h \in S_n$ are conjugate iff they have the same cycle structure.

Solution.

(a) Let $c := (a_1, a_2, \ldots, a_k)$. To show that fcf^{-1} and $g := (f(a_1), f(a_2), \ldots, f(a_k))$ are the same functions, we check that $fcf^{-1}(x) = g(x)$ for all $x \in \{1, \ldots, n\}$.

Case 1, $x \in \{f(a_1), f(a_2), \ldots, f(a_k)\}$: Suppose $x = f(a_i)$. Then

$$fcf^{-1}(x) = fcf^{-1}f(a_i) = fc(a_i) = \begin{cases} f(a_{i+1}) & \text{if } i < k \\ f(a_1) & \text{if } i = k \end{cases}$$

For $g(x)$ we get the same result.

Case 2, $x \in \{1, \ldots, n\} \setminus \{f(a_1), f(a_2), \ldots, f(a_k)\}$: Then $f^{-1}(x) \notin \{a_1, a_2, \ldots, a_k\}$ and $f^{-1}(x)$ is fixed by c. So

$$fcf^{-1}(x) = ff^{-1}(x) = x$$

For $g(x)$ we get the same result.

(b) Define $f \in S_n$ by $f(a_1) := b_1, \ldots, f(a_k) := b_k$ and as an arbitrary bijection from $\{1, \ldots, n\} \setminus \{a_1, a_2, \ldots, a_k\}$ to $\{1, \ldots, n\} \setminus \{b_1, b_2, \ldots, b_k\}$. □

(5) (a) How many different conjugacy classes are there in S_4?

(b) How many distinct 3-cycles are there in S_4?

(c) How many elements in S_4 are conjugate to $(1\ 2)(3\ 4)$?

Hint: Use (4).

Solution. (a) By (4) the number of conjugacy classes of S_4 is equal to the number of different cycle structures of elements in S_4, which is just the number of partitions of 4:

$$\begin{aligned} 4, &\text{ class of } (1234) \\ 3+1, &\text{ class of } (123) \\ 2+2, &\text{ class of } (12)(34) \\ 2+1+1, &\text{ class of } (12) \\ 1+1+1+1, &\text{ class of } () \end{aligned}$$

Hence there are 5 conjugacy classes.
(b) For a 3-cycle $(a\ b\ c)$ in S_4, we have 4 options to pick a, then 3 to pick b, finally 2 to pick c. For this, there are $4 \cdot 3 \cdot 2$ choices. However every 3-cycle can be written in 3 distinct ways,

$$(a\ b\ c) = (b\ c\ a) = (c\ a\ b).$$

Hence we have to divide the number above by 3 and get $\frac{4 \cdot 3 \cdot 2}{3} = 8$ distinct 3-cycles. The conjugacy class of (123) has size 8.
(c) By (4) the conjugacy class of $g = (1\ 2)(3\ 4)$ consists of all products of two disjoint transpositions $(a\ b)(c\ d)$. What are they? There are 3 options for putting an element in $\{2, 3, 4\}$ in the same transposition as 1; then the other 2 elements will go into the second transposition (order does not matter). So the conjugacy class of g is

$$\{(1\ 2)(3\ 4),\ (1\ 3)(2\ 4),\ (1\ 4)(2\ 3)\}.$$

□

(6) **Normal subgroups are kernels:** Let N be normal in G. Show that N is the kernel of the homomorphism

$$G \to G/N, x \mapsto xN.$$

Solution. Proof of Thm 10.4 in Gallian's book. □

(7) **Correspondence Theorem between normal subgroups:** Let $\varphi\colon G \to H$ be an onto homomorphism. Show
   (a) If B is normal in H, then $\varphi^{-1}(B)$ is normal in G and contains $\ker \varphi$.
   (b) If A is normal in G, then $\varphi(A)$ is normal in H.

Solution. (a) Assume B is normal in H. Let $x \in \varphi^{-1}(B)$ and $g \in G$. To show $gxg^{-1} \in \varphi^{-1}(B)$ consider

$$\varphi(gxg^{-1}) = \varphi(g)\varphi(x)\varphi(g)^{-1}$$

which is in B because $\varphi(x)$ is in B and B is normal. Hence $\varphi^{-1}(B)$ is normal.

(b) is similar.