Math 3140 - Assignment 9

Due October 23, 2026

(1) Determine the kernels and images of the following homomorphisms. Which are injective, surjective?
   (a) $\varphi\colon \mathbb{Z}_6 \to \mathbb{Z}_6,\ x \mapsto 4x$
   (b) $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2,\ \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$
   for the additive group $\mathbb{R}^2$.
   (c) $h\colon \mathbb{C}^* \to \mathbb{C}^*, x \mapsto x^2$, where $\mathbb{C}^*$ denotes the multiplicative group of the complex numbers without 0.

(2) Let $n \in \mathbb{N}$. Show that $\operatorname{sign}\colon S_n \to (\{-1, 1\}, \cdot)$ defined by

$$\operatorname{sign}(f) := \frac{\prod_{1 \le i < j \le n}(f(j) - f(i))}{\prod_{1 \le i < j \le n}(j - i)}$$

is a homomorphism.
Hint: Use that for $f, g \in S_n$ and $1 \le i < j \le n$

$$\frac{fg(j) - fg(i)}{g(j) - g(i)} = \frac{fg(i) - fg(j)}{g(i) - g(j)}.$$

Hence you can reorder these factors such that the denominator is always positive.

(3) Show $\operatorname{sign}(f) = -1$ for any transposition $f = (a\ b)$ in S_n.
Hint: Count the number of inversions of f, that is, the pairs $1 \le x < y \le n$ such that $f(x) > f(y)$. Recall from class that $\operatorname{sign}(f) = (-1)^{\text{number of inversions of } f}$.

(4) **When are two elements of S_n conjugate?**
   (a) Show that for any k-cycle $(a_1, a_2, \ldots, a_k) \in S_n$ and any $f \in S_n$, we have

$$f(a_1, a_2, \ldots, a_k)f^{-1} = (f(a_1), f(a_2), \ldots, f(a_k)).$$

   (b) For any two k-cycles $(a_1, a_2, \ldots, a_k), (b_1, b_2, \ldots, b_k) \in S_n$ explicitly give $f \in S_n$, such that

$$f(a_1, a_2, \ldots, a_k)f^{-1} = (b_1, b_2, \ldots, b_k).$$

The *cycle structure* of a permutation g is the length of the cycles in the cycle decomposition of g (counted with multiplicity). For example $g = (1\ 2\ 3)(4\ 5)(6\ 7)$ has cycle structure $3, 2, 2$.

Deduce that two permutations $g, h \in S_n$ are conjugate iff they have the same cycle structure.

(5) (a) How many different conjugacy classes are there in S_4?

(b) How many distinct 3-cycles are there in S_4?
(c) How many elements in S_4 are conjugate to $(1\ 2)(3\ 4)$?

Hint: Use (4).

(6) **Normal subgroups are kernels:** Let N be normal in G. Show that N is the kernel of the homomorphism

$$G \to G/N, x \mapsto xN.$$

(7) **Correspondence Theorem between normal subgroups:** Let $\varphi\colon G \to H$ be an onto homomorphism. Show
(a) If B is normal in H, then $\varphi^{-1}(B)$ is normal in G and contains $\ker \varphi$.
(b) If A is normal in G, then $\varphi(A)$ is normal in H.