Math 3140 - Assignment 8

Due October 16, 2026

(1) (a) Show that a subgroup N of G is normal iff N is a union of conjugacy classes of G.
(b) Which conjugacy classes are contained in the center $Z(G)$?

Solution.
(a) Assume $N \trianglelefteq G$. Let $x \in N$. By normality every conjugate gxg^{-1} of x for $g \in G$ is in N. So N contains the whole conjugacy class of x. Hence N is the union of the conjugacy classes of its elements.
Conversely, assume $N \leq G$ is a union of conjugacy classes. Then for all $x \in N, g \in G$ we have $gxg^{-1} \in N$. Thus N is normal.
(b) If $x \in Z(G)$, then $x = gxg^{-1}$ is only conjugate to itself. So the conjugacy class of x is $\{x\}$. Hence $Z(G)$ is the union of the 1-element conjugacy classes of G. □

(2) Use (1) to determine all normal subgroups of S_3.
(a) What is $Z(S_3)$?
(b) Describe the quotient groups S_3/N for these normal subgroups up to isomorphism as simple as possible.

Solution. By computation we see that the conjugacy classes of S_3 are

$$\{()\} \quad \{(123), (132)\} \quad \{(12), (13), (23)\}.$$

Which unions of them give subgroups? Only

$$\{()\} \quad A_3 = \{(), (123), (132)\} \quad S_3.$$

(a) The center is the union of the 1-element classes, so $\{()\}$.
(b)

$$S_3/\{()\} \cong S_3 \quad S_3/A_3 \cong \mathbb{Z}_2 \quad S_3/S_3 \cong 1.$$

(3) Show that every subgroup H of index 2 in a group G is normal. Hint: Look at the partition of G into cosets of H.

Solution. Since G has 2 left cosets by assumption, $G = H \cup xH$ for any $x \in G \setminus H$. Note that $H \neq Hx$ for $x \in G \setminus H$ as well. Thus $G = H \cup Hx$.

Since these unions are disjoint we get $xH = G \setminus H = Hx$. Hence the left cosets of and right cosets of H in G are the same. H is normal in G. □

(4) Let N be a normal subgroup of G. Show
(a) If G is abelian, then G/N is abelian.

(b) If G is cyclic, then G/N is cyclic.
(c) Give a nonabelian group G with G/N and N abelian.

Solution.
(a) Assume G is abelian. For $x, y \in G$

$$xNyN = xyN = yxN = yNxN$$

So G/N is abelian
(b) Let $G = \langle a \rangle$. Then $G/N = \langle aN \rangle$.
(c) By (2b) S_3 is nonabelian but has a cyclic normal subgroup A_3 and quotient $S_3/A_3 \cong \mathbb{Z}_2$. □

(5) Determine the orders of the following quotient groups and whether they are cyclic.
(a) $\mathbb{Z}/\langle 10 \rangle$
(b) $\mathbb{Z}_{12}/\langle 6 \rangle$
(c) $\mathbb{Z} \times \mathbb{Z}/\langle (2,3) \rangle$
(d) $\mathbb{Z}_4 \times \mathbb{Z}_6 \, / \langle (2,2) \rangle$

Solution.
(a) $\mathbb{Z}/\langle 10 \rangle = \mathbb{Z}_{10}$ is cyclic of order 10.
(b) $\mathbb{Z}_{12}/\langle 6 \rangle \cong \mathbb{Z}_6$ is cyclic of order 6.
(c) $\mathbb{Z}\times\mathbb{Z}/\langle (2,3) \rangle$ is infinite and generated by the coset of $(1,1)$. Note that $\langle (1,1), (2,3) \rangle = \mathbb{Z}^2$ since $(1,0) = 3(1,1) - (2,3)$ and $(0,1) = -2(1,1) + (2,3)$. Since every element of $\mathbb{Z}^2$ is of the form $a(1,1)+b(2,3)$ for some $a, b \in \mathbb{Z}$, every coset of $N := \langle (2,3) \rangle$ has the form $a(1,1) + N$. Hence the quotient is generated by $(1,1) + N$.
(d) $\mathbb{Z}_4 \times \mathbb{Z}_6 \, / \langle (2,2) \rangle$ has order 4 since $N := \langle (2,2) \rangle = \{(0,0), (2,2), (0,4), (2,0), (0,2), (2,4)\}$. The cosets of N are $N, (1,0) + N, (0,1) + N, (1,1) + N$. Since each of these elements has order ≤ 2 in the quotient, $\mathbb{Z}_4 \times \mathbb{Z}_6 \, / \langle (2,2) \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is not cyclic. □

(6) Show that $\pi \colon G \times H \to G, \ (g,h) \to g$ is a homomorphism. Determine its kernel and image. Show that $G \times H/\{1\} \times H \cong G$.

Solution. $\ker \pi = 1 \times H$ and the image is G. By the first isomorphism theorem $(G \times H)/(\{1\} \times H) \cong G$. □

(7) For a subgroup H and a normal subgroup N of G show that

$$HN := \{hn \ : \ h \in H, n \in N\}$$

is a subgroup of G.

Solution.
- $1 \in HN \neq \emptyset$.

- Let $h_1, h_2 \in H, n_1, n_2 \in N$. Since N is normal,
$$(h_1 n_1)(h_2 n_2) = h_1 h_2 (h_2^{-1} n_1 h_2) n_2 \in HN.$$
- Let $h \in H, n \in N$. Then $(hn)^{-1} = n^{-1}h^{-1} = h^{-1}(hn^{-1}h^{-1}) \in HN$. □

(8) **Characterization of direct products:**

(a) Let $G = K \times N$ be an external direct product. Show that $K \times 1$ and $1 \times N$ are normal subgroups of G such that
$$(K \times 1) \cap (1 \times N) = 1 \quad \text{and} \quad (K \times 1)(1 \times N) = G.$$

(b) Let G be a group with normal subgroups K, N such that
$$K \cap N = 1 \quad \text{and} \quad KN = G.$$
Then G is called an **internal direct product** of K and N. Show that
$$G \cong K \times N.$$
Hint: Show that $\varphi\colon K \times N \to G, (k, n) \mapsto kn$, is an isomorphism.

Solution.

(a) $K \times 1 \trianglelefteq G$ since $(x, y)^{-1}\, K \times 1\, (x, y) = x^{-1}Kx \times 1 = K \times 1$ for all $(x, y) \in G$. Similarly $1 \times N \trianglelefteq G$. $(K \times 1) \cap (1 \times N) = 1$ and $(K \times 1)(1 \times N) = G$ are clear.

(b) $\varphi\colon K \times N \to G, (k, n) \mapsto kn$, is
- surjective by the assumption $KN = G$;
- injective: for $(x_1, y_1), (x_2, y_2) \in K \times N$, $x_1 y_1 = x_2 y_2$ yields $x_2^{-1} x_1 = y_2 y_1^{-1} \in K \cap N = 1$;
- a homomorphism: for $(x_1, y_1), (x_2, y_2) \in K \times N$,
$$\varphi((x_1, y_1)(x_2, y_2)) = x_1 x_2 y_1 y_2$$
$$\varphi(x_1, y_1)\varphi(x_2, y_2) = x_1 y_1 x_2 y_2$$
are equal since any $x \in K$ commutes with any $y \in N$:
$$\underbrace{xyx^{-1}}_{\in N} y^{-1} = x \underbrace{yx^{-1}y^{-1}}_{\in K} \in K \cap N = 1.$$
Hence $xy = yx$. □