

Math 3140 - Assignment 8

Due October 17, 2026

- (1) (a) Show that a subgroup N of G is normal iff N is a union of conjugacy classes of G .
(b) Which conjugacy classes are contained in the center $Z(G)$?
- (2) Use (1) to determine all normal subgroups of S_3 .
(a) What is $Z(S_3)$?
(b) Describe the quotient groups S_3/N for these normal subgroups up to isomorphism as simple as possible.
- (3) Show that every subgroup H of index 2 in a group G is normal.
Hint: Look at the partition of G into cosets of H .
- (4) Let N be a normal subgroup of G . Show
(a) If G is abelian, then G/N is abelian.
(b) If G is cyclic, then G/N is cyclic.
(c) Give a nonabelian group G with G/N and N abelian.
- (5) Determine the orders of the following quotient groups and whether they are cyclic.
(a) $\mathbb{Z}/\langle 10 \rangle$
(b) $\mathbb{Z}_{12}/\langle 6 \rangle$
(c) $\mathbb{Z} \times \mathbb{Z}/\langle (2, 3) \rangle$
(d) $\mathbb{Z}_4 \times \mathbb{Z}_6 / \langle (2, 2) \rangle$
- (6) Show that $\pi: G \times H \rightarrow G, (g, h) \rightarrow g$ is a homomorphism.
Determine its kernel and image.
Show that $G \times H/\{1\} \times H \cong G$.
- (7) For a subgroup H and a normal subgroup N of G show that

$$HN := \{hn : h \in H, n \in N\}$$

is a subgroup of G .

- (8) **Characterization of direct products:**
(a) Let $G = K \times N$ be an external direct product. Show that $K \times 1$ and $1 \times N$ are normal subgroups of G such that
 $(K \times 1) \cap (1 \times N) = 1$ and $(K \times 1)(1 \times N) = G$.
(b) Let G be a group with normal subgroups K, N such that

$$K \cap N = 1 \quad \text{and} \quad KN = G.$$

Then G is called an **internal direct product** of K and N .
Show that

$$G \cong K \times N.$$

Hint: Show that $\varphi: K \times N \rightarrow G, (k, n) \mapsto kn$, is an isomorphism.