

Math 3140 - Assignment 7

Due October 9, 2026

- (1) The **exponent** of a group G is the smallest $n > 0$ such that $g^n = 1$ for all $g \in G$ if it exists; else the exponent of G is infinite. Show that every group G of exponent 2 is abelian.

Solution Let G have exponent 2, let $a, b \in G$. Then

$$ab = (ab)^{-1} = b^{-1}a^{-1} = ba$$

and G is abelian.

- (2) Let $(G, \cdot)$ be a group and $S \subseteq G$. Show that the subgroup generated by S consists of all finite products of integer powers of elements in S , i.e.

$$\langle S \rangle = \{a_1^{k_1} \cdots a_n^{k_n} : n \geq 0, k_1, \dots, k_n \in \mathbb{Z}, a_1, \dots, a_n \in S\}$$

Hint: Show that if a subgroup H contains S , then it also contains all products of powers of elements in S .

Conversely, show that the set of products of powers of elements in S is a subgroup of G .

Solution Let $H := \{a_1^{k_1} \cdots a_n^{k_n} : n \geq 0, k_1, \dots, k_n \in \mathbb{Z}, a_1, \dots, a_n \in S\}$.

$\supseteq$: Since $\langle S \rangle$ is closed under multiplication and inverses, we have

- $a \in S \Rightarrow a^k \in \langle S \rangle$ for all $k \in \mathbb{Z}$ and further
- $a_1, \dots, a_k \in S \Rightarrow a_1^{k_1} \cdots a_n^{k_n} \in \langle S \rangle$ for all $k_1, \dots, k_n \in \mathbb{Z}$.

Hence $\langle S \rangle \supseteq H$.

$\subseteq$: Since $\langle S \rangle$ is the smallest subgroup containing S , it suffices to show that H is a subgroup that contains S :

- $S \subseteq H$ is clear.
- For $a_1^{k_1} \cdots a_n^{k_n}, b_1^{\ell_1} \cdots b_m^{\ell_m} \in H$ we see that $a_1^{k_1} \cdots a_n^{k_n} b_1^{\ell_1} \cdots b_m^{\ell_m} \in H$.
- For $a_1^{k_1} \cdots a_n^{k_n} \in H$ we see that $(a_1^{k_1} \cdots a_n^{k_n})^{-1} = a_n^{-k_n} \cdots a_1^{-k_1}$ is also in H .

Hence H is a subgroup and contains $\langle S \rangle$.

- (3) For G a permutation group on X and $x \in X$, the **stabilizer** of x in G is

$$\text{stab}_G(x) := \{g \in G : g(x) = x\},$$

the set of all permutations that fix x .

Show that $\text{stab}_G(x)$ is a subgroup of G .

Solution $\text{stab}_G(x)$ is a subgroup since it's non-empty, closed under multiplication and inverses:

- (a) $() \in \text{stab}_G(x) \neq \emptyset$,
 - (b) for $f, g \in \text{stab}_G(x)$ we see $f(g(x)) = x$ and $f \circ g \in \text{stab}_G(x)$,
 - (c) for $f \in \text{stab}_G(x)$ we see $f^{-1}(x) = x$ and $f^{-1} \in \text{stab}_G(x)$.
- (4) For the group $G = \{(), (12)(34), (1234)(56), (13)(24), (1432)(56), (56)(13), (14)(23), (24)(56)\}$:
- (a) Find the stabilizer of 1 and the orbit of 1.
 - (b) Find the stabilizer of 3 and the orbit of 3.
 - (c) Find the stabilizer of 5 and the orbit of 5.

Solution

- (a) $\text{stab}_G(1) = \{(), (24)(56)\}$ and $G(1) = \{1, 2, 3, 4\}$.
 - (b) $\text{stab}_G(3) = \{(), (24)(56)\}$ and $G(3) = \{1, 2, 3, 4\}$.
 - (c) $\text{stab}_G(5) = \{(), (12)(34), (13)(24), (14)(23)\}$ and $G(5) = \{5, 6\}$.
- (5) Give an isomorphism between the following pairs of groups or explain why they are not isomorphic.
- (a) $\mathbb{Z}_8 \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_4$
 - (b) $\mathbb{Z}_{12}, \mathbb{Z}_4 \times \mathbb{Z}_3$
 - (c) $\mathbb{Z}_4 \times \mathbb{Z}_2, D_8$

Solution.

- (a) $\mathbb{Z}_8 \times \mathbb{Z}_2 \not\cong \mathbb{Z}_4 \times \mathbb{Z}_4$ since $(1, 0) \in \mathbb{Z}_8 \times \mathbb{Z}_2$ has order 8 but all elements of $\mathbb{Z}_4 \times \mathbb{Z}_4$ have order dividing 4
 - (b) $\mathbb{Z}_{12} \rightarrow \mathbb{Z}_4 \times \mathbb{Z}_3, [x]_{12} \mapsto ([x]_3, [x]_4)$ is an isomorphism.
 - (c) $\mathbb{Z}_4 \times \mathbb{Z}_2 \not\cong D_8$ since the first is abelian, the second not
- (6) Let p be a prime. Show that $\mathbb{Z}_p \times \mathbb{Z}_p$ has exactly $p+1$ subgroups of order p .

Give one generator for each subgroup of order p .

Solution. Every nonzero element in $\mathbb{Z}_p \times \mathbb{Z}_p$ has order p . So there are $p^2 - 1$ elements of order p , each of which generates a subgroup of order p . However any subgroup of order p has $p - 1$ distinct generators. So the number of distinct subgroups of order p is $\frac{p^2-1}{p-1} = p + 1$.

Note that every single element $(1, a)$ for $a \in \mathbb{Z}_p$ and $(0, 1)$ generate distinct subgroups of order p . So these $p + 1$ elements generate all the $p + 1$ distinct subgroups of order p . $\square$

- (7) For a group G and $x, y \in G$ say x is **conjugate** to y in G if

$$\exists g \in G : y = gxg^{-1}.$$

Show that conjugacy is an equivalence relation.

Its equivalence classes are called the **conjugacy classes** of G .

Solution. Show that conjugacy is reflexive, symmetric and transitive.

reflexive: Let $x \in G$. Then x is conjugate to itself because $x = 1x1^{-1}$ (so 1 conjugates x to x)

symmetric: Let $x, y \in G$. Assume x is conjugate to y , that is, $y = gxg^{-1}$ for some $g \in G$ (so g conjugates x to y). Then $g^{-1}yg = x$, which means that g^{-1} conjugates y to x ; so y is conjugate to x .

transitive: Let $x, y, z \in G$. Assume x is conjugate to y and y is conjugate to z , that is, $y = gxg^{-1}$ and $z = hyh^{-1}$ for some $g, h \in G$. Then $z = hgx(hg)^{-1}$, which means that z is conjugate to x .

Thus conjugacy is an equivalence relation.