

Math 3140 - Assignment 6

Due October 2, 2026

- (1) For the following subgroups H of G , find all the left cosets of H in G . Give one representative for each left coset. How many are there?
- (a) $G = \mathbb{R}^2$ under addition, $H = \{(x, 0) : x \in \mathbb{R}\} = \mathbb{R} \times \{0\}$
 - (b) $G = \langle a \rangle$ of order 12, $H = \langle a^4 \rangle$
 - (c) $G = \mathbb{R}^*$ under multiplication, $H = \mathbb{R}^+$ the subgroup of positive reals

Solution.

- (a) Cosets are $(0, y) + H$ for $y \in \mathbb{R}$; all the translates (parallels) of the x -axis H .
 - (b) $H = \{1, a^4, a^8\}$ has size 3 and index 4 in G . Cosets are H, aH, a^2H, a^3H .
 - (c) $\mathbb{R}^* = H \cup (-1)H$, hence $[G : H] = 2$.
- (2) Let G be a finite group with subgroups $H \leq K \leq G$. Show that

$$|G : H| = |G : K| \cdot |K : H|.$$

Solution. Recall that $|G : H| = \frac{|G|}{|H|}$, etc. So

$$|G : K| \cdot |K : H| = \frac{|G|}{|K|} \cdot \frac{|K|}{|H|} = \frac{|G|}{|H|} = |G : H|.$$

□

- (3) For groups $(G, \cdot_G)$ and $(H, \cdot_H)$ show that the *direct product*

$$G \times H := \{(g, h) : g \in G, h \in H\}$$

under the operation

$$(g_1, h_1) \cdot (g_2, h_2) := (g_1 \cdot_G g_2, h_1 \cdot_H h_2)$$

for $g_1, g_2 \in G, h_1, h_2 \in H$ is a group.

Solution.

- (a) Clearly $\cdot$ is a function from $(G \times H)^2$ to $G \times H$.
- (b) $\cdot$ inherits associativity from the multiplications on G and on H .
- (c) $(1_G, 1_H)$ is the identity of $G \times H$ where $1_G, 1_H$ is the identity of G, H , respectively.
- (d) Any $(g, h) \in G \times H$ has an inverse $(g, h)^{-1} = (g^{-1}, h^{-1})$.

□

- (4) Give an isomorphism between the following pairs of groups or explain why they are not isomorphic.
- (a) $G \times H, H \times G$
 - (b) $\mathbb{C}, \mathbb{R} \times \mathbb{R}$ under addition
 - (c) $\mathbb{C}^*, \mathbb{R}^* \times \mathbb{R}^*$ under multiplication
 - (d) $\mathbb{Z} \times \mathbb{Z}, (\{2^x 3^y : x, y \in \mathbb{Z}\}, \cdot)$

Solution.

- (a) $G \times H \times H \times G, (g, h) \mapsto (h, g)$, is an isomorphism.
 - (b) $\mathbb{C} \rightarrow \mathbb{R} \times \mathbb{R}, a + bi \mapsto (a, b)$, is an isomorphism since addition of complex numbers is componentwise.
 - (c) $\mathbb{C}^* \not\cong \mathbb{R}^* \times \mathbb{R}^*$ since e.g. -1 is the unique element of order 2 in $\mathbb{C}^*$ but $(-1, 1), (1, -1), (-1, -1)$ all have order 2 in $\mathbb{R}^* \times \mathbb{R}^*$
 - (d) $\mathbb{Z} \times \mathbb{Z} \rightarrow (\{2^x 3^y : x, y \in \mathbb{Z}\}, \cdot), (x, y) \mapsto 2^x 3^y$, is an isomorphism.
- (5) Note that $1 \times H$ is a subgroup of $G \times H$. Find all the left cosets of $1 \times H$ in $G \times H$. Give one representative for each left coset. How many are there?

Solution. Every left coset of $1 \times H$ in $G \times H$ is of the form $\{g\} \times H$ for $g \in G$.

So $(g, 1)$ for $g \in G$ is a set of representatives through all cosets. There are as many as elements in G . $\square$

- (6) Let G be a nontrivial group that has no proper, nontrivial subgroups (i.e. 1 and G are the only subgroups of G). Show that $|G|$ is prime.

Hint: Do not assume at the outset that G is finite.

Solution: Let $a \in G, a \neq 1$. Then $\langle a \rangle$ is a nontrivial subgroup of G . Hence $\langle a \rangle = G$ is cyclic.

Now G cannot be infinite because otherwise $\langle a^2 \rangle$ is a proper nontrivial subgroup.

So G is finite, say of order n . For $d|n$, the subgroup $\langle a^d \rangle$ must either be 1 or G . In the former case $d = n$, in the latter $d = 1$. Anyways, n is only divisible by 1 and by n . Thus n is prime. $\square$