

Math 3140 - Assignment 5

Due September 25, 2026

- (1) Let $\mathbb{C}$ be the set of complex numbers and

$$M = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

Show that $(\mathbb{C}, +) \cong (M, +)$ and $(\mathbb{C} \setminus \{0\}, \cdot) \cong (M \setminus \{0\}, \cdot)$.

Solution: We need to define the isomorphisms.

$$\varphi: \mathbb{C} \rightarrow M, a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

is clearly bijective.

- (a) Check $\varphi([a + ib] + [c + id]) = \varphi(a + ib) + \varphi(c + id)$ for all $a + ib, c + id \in \mathbb{C}$.

Hence $\varphi: (\mathbb{C}, +) \rightarrow (M, +)$ is an isomorphism.

- (b) Check $\varphi([a + ib] \cdot [c + id]) = \varphi(a + ib) \cdot \varphi(c + id)$ for all $a + ib, c + id \in \mathbb{C}$.

Hence $\varphi: (\mathbb{C} \setminus \{0\}, \cdot) \rightarrow (M \setminus \{0\}, \cdot)$ is an isomorphism. $\square$

- (2) Let G be a group. Show that $\text{Aut}G$ is group a under composition of functions. **Solution:** Check the definition of a group:

- (a) $\text{Aut}G \neq \emptyset$ since the identity map id is in $\text{Aut}G$.

- (b) Composition is an operation on $\text{Aut}G$ since the composition of homomorphisms is a homomorphism and the composition of bijections is a bijection again.

- (c) Composition of function is associative (recall from Discrete Math, Calculus).

- (d) There is an identity element: $\text{id} \circ \varphi = \varphi \circ \text{id} = \varphi$ for all $\varphi \in \text{Aut}G$.

- (e) Every $\varphi \in \text{Aut}G$ has an inverse $\varphi^{-1} \in \text{Aut}G$.

Hence $(\text{Aut}G, \circ)$ is a group. $\square$

- (3) For a group G and $g \in G$, define the inner automorphism

$$\varphi_g: G \rightarrow G, x \mapsto gxg^{-1}.$$

Show

- (a) $\varphi_g \in \text{Aut}G$.

- (b) $\Phi: G \rightarrow \text{Aut}G, g \mapsto \varphi_g$, is a homomorphism.

- (c) $\ker \Phi = Z(G)$.

Solution: (a) φ_g is a homomorphism since for $x, y \in G$

$$\varphi_g(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1} = \varphi_g(x)\varphi_g(y).$$

φ_g is bijective since it has an inverse $\varphi_g^{-1} = \varphi_{g^{-1}}$. Thus $\varphi_g \in \text{Aut}G$.

(b) Φ is a homomorphism since for $g, h \in G$

$$\Phi(gh) = \varphi_{gh} \text{ maps } x \mapsto ghx(gh)^{-1},$$

$$\Phi(g)\Phi(h) = \varphi_g \circ \varphi_h \text{ maps } x \mapsto \varphi_g(\varphi_h(x)) = ghx(gh)^{-1}.$$

$$\text{Hence } \Phi(gh) = \Phi(g)\Phi(h).$$

(c)

$$g \in \ker \Phi \text{ iff } \varphi_g = \text{id}$$

$$\text{iff } gxg^{-1} = x \quad \forall x \in G$$

$$\text{iff } g \in Z(G).$$

□

(4) Let $D_8 = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$ be the dihedral group of order 8 generated by a rotation a and a reflection b .

(a) When are $\varphi_g = \varphi_h$ for $g, h \in D_8$?

(b) What is the order of $\text{Inn}D_8$?

(c) List all distinct elements in $\text{Inn}D_8$.

Solution: (a) For any group G and $g, h \in G$:

$$\varphi_g = \varphi_h \text{ iff } \varphi_g(x) = \varphi_h(x) \quad \forall x \in G$$

$$\text{iff } gxg^{-1} = h x h^{-1} \quad \forall x \in G$$

$$\text{iff } xg^{-1}h = g^{-1}hx \quad \forall x \in G$$

$$\text{iff } g^{-1}h \in Z(G)$$

$$\text{iff } gZ(G) = hZ(G).$$

Hence g, h give the same inner automorphism iff they are in the same left coset of the center of G .

(b) By (a)

$$|\text{Inn}G| = \frac{|G|}{|Z(G)|}.$$

Since $Z(D_8) = \{1, a^2\}$, we get $|\text{Inn}D_8| = 4$.

(b) By (a) we need to pick one representative from each coset of the center to get all distinct elements in $\text{Inn}G$. So

$$\text{Inn}D_8 = \{\varphi_1 = \varphi_{a^2}, \varphi_a = \varphi_{a^3}, \varphi_b = \varphi_{a^2b}, \varphi_{ab} = \varphi_{a^3b}\}.$$

- (5) Show that $|\text{Aut}D_8| \leq 8$.

Hint: Explain why an automorphism is uniquely determined by what it does to the generators a and b . Use the properties of automorphisms to restrict where could these be mapped to.

Solution: Recall that D_8 is generated by a rotation a (order 4) and a reflection b (order 2),

$$D_8 = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}.$$

If a homomorphism $\varphi: D_8 \rightarrow H$ satisfies $\varphi(a) = u$ and $\varphi(b) = v$, then the homomorphism property yields that $\varphi(a^i b^j) = u^i v^j$. Hence φ is uniquely determined on the whole group by what it does to its generators.

Now let $\varphi \in \text{Aut}D_8$. Isomorphisms preserve the order of elements and map generators to generators. Since $\varphi(a)$ has order 4, it can only be a or a^3 . Further $\varphi(b)$ has order 2, so could be a^2, b, ab, a^2b, a^3b . But $\varphi(b) = a^2$ is not possible since then $\langle \varphi(a), \varphi(b) \rangle = \langle a \rangle \neq D_8$. So it only remains that $\varphi(b) \in \{b, ab, a^2b, a^3b\}$.

Summing up there are 2 choices for $\varphi(a)$ and 4 choices for $\varphi(b)$. Hence at most 8 automorphism. (Note: It's not clear from this argument that every choice really yields an automorphism but it does.) $\square$

- (6) Find non-isomorphic groups G, H such that $\text{Aut}G \cong \text{Aut}H$.

Solution: We classified $\text{Aut}\mathbb{Z}_n$ in class. So consider

$$\text{Aut}\mathbb{Z}_3 \cong (\mathbb{Z}_3^*, \cdot) = (\{1, 2\}, \cdot)$$

$$\text{Aut}\mathbb{Z}_4 \cong (\mathbb{Z}_4^*, \cdot) = (\{1, 3\}, \cdot)$$

Both groups are cyclic of order 2, hence are isomorphic to $(\mathbb{Z}_2, +)$. $\square$

- (7) Let $(G, \cdot)$ be a group and $g \in G$. Show that the translation

$$a_g: G \rightarrow G, x \mapsto gx,$$

is a permutation on G . Is a_g a homomorphism on G ?

Solution: The translation a_g by g is bijective because it has an inverse, namely $a_{g^{-1}}$, the translation by g^{-1} .

Since $a_g(1) = g1 = g$, we see that a_g is not a homomorphism if $g \neq 1$. But $a_1 = \text{id}_G$ is a homomorphism. $\square$