# Math 3140 - Assignment 4

Due September 18, 2026

(1) For permutations $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{pmatrix}$ compute

$$\alpha^{-1}, \alpha\beta, \beta\alpha$$

and their orders.

**Solution:** Rewrite in cycle notation

$$\alpha = (1\,2)(4\,5), \quad \beta = (1\,6\,5\,3\,2)$$

Then the inverse of $\alpha$ can be found by reversing its cycles

$$\alpha^{-1} = (2\,1)(5\,4) = (1\,2)(4\,5) = \alpha,$$

has order 2.

$$\alpha\beta = (1\,6\,4\,5\,3)$$
$$\beta\alpha = (2\,6\,5\,4\,3)$$

both have order 5 (but in general $|\alpha\beta| \neq |\beta\alpha|$).

(2) Find all subgroups of $S_3$, give generators for each and show their inclusions in a subgroup lattice.

Hint: First find all cyclic subgroups. Then all subgroups that need two generators, etc.

**Solution:** trivial subgroup: 1

cyclic subgroups: $\langle(12)\rangle$ $\langle(13)\rangle$ $\langle(23)\rangle$ $\langle(123)\rangle$

any 2 distinct non-identity permutations generate $S_3$ $\square$

(3) Show that a cycle of length $\ell$ can be written as a product of $\ell - 1$ transpositions,

$$(a_1\ a_2\ \ldots a_\ell) = (a_1\ a_\ell)(a_1 a_{\ell-1}) \ldots (a_1 a_2).$$

**Solution:** Note that the functions on each side map

$a_1 \mapsto a_2$
$a_2 \mapsto a_3$
$\vdots$
$a_\ell \mapsto a_1$

and fix all other points. Hence they are equal. $\square$

(4) Show that $Z(S_n) = 1$ for $n \geq 3$.

**Solution:** $\supseteq$: Clearly the identity () commutes with all elements in $S_n$, hence is in its center.

$\subseteq$: Let $\alpha \in S_n$ such that $\alpha \neq ()$. Show that there is some $\beta \in S_n$ such that $\alpha\beta \neq \beta\alpha$. For that $\beta$ needs to move the same points as $\alpha$ but differently.

Since $\alpha \neq ()$, we have $x \in \{1, \dots, n\}$ such that $\alpha(x) \neq x$. Let $y \in \{1, \dots n\}$ be distinct from $x$ and $\alpha(x)$. Such $y$ exists since $n \geq 3$. Let $\beta := (x, y)$.

Then $\alpha\beta(x) = \alpha(y)$ and $\beta\alpha(x) = \alpha(x)$. Note $\alpha(x) \neq \alpha(y)$ since $x \neq y$. So $\alpha\beta \neq \beta\alpha$ and $\alpha \notin Z(S_n)$. Thus $Z(S_n) \subseteq 1$. $\square$

(5) Let $(G, \cdot, ^{-1}, 1_G)$ and $(H, *, ^{-1}, 1_H)$ be groups with identities $1_G, 1_H$, respectively. Let $\varphi \colon G \to H$ satisfy

$$\varphi(x \cdot y) = \varphi(x) * \varphi(y) \text{ for all } x, y \in G.$$

Show:

(a) $\varphi(1_G) = 1_H$.
Hint: Start by evaluating $\varphi(1_G \cdot 1_G)$ in two ways.

(b) $\varphi(x^{-1}) = \varphi(x)^{-1}$ for all $x \in G$.
Hint: Use (a).

**Solution:** (a)

$$\varphi(1_G) = \varphi(1_G \cdot 1_G) = \varphi(1_G) * \varphi(1_G)$$

Multiply by $\varphi(1_G)^{-1}$ in $H$ to get $1_H = \varphi(1_G)$.

(b) $\varphi(g^{-1}) \cdot \varphi(g) = \varphi(g^{-1}g) = \varphi(1_G) = 1_H$ by (a) yields that $\varphi(g^{-1})$ is the inverse of $\varphi(g)$. $\square$

(6) Let $G, H$ be groups and $\varphi \colon G \to H$ a homomorphism. Show

(a) The image of $\varphi$,

$$\varphi(G) := \{\varphi(g) \ : \ g \in G\}$$

is a subgroup of $H$.

(b) The kernel of $\varphi$,

$$\ker \varphi := \{g \in G \ : \ \varphi(g) = 1\}$$

is a subgroup of $G$.

Recall image and kernel of linear maps on vector spaces.

**Solution:** (a) Show that $\varphi(G)$ is closed under multiplication and inverses:

Let $a, b \in \varphi(G)$. Then we have $x, y \in G$ such that $a = \varphi(x), b = \varphi(y)$. Now

$$ab = \varphi(x)\varphi(y) = \varphi(xy) \in \varphi(G),$$

$$a^{-1} = \varphi(x^{-1}) \in \varphi(G) \text{ by (5a).}$$

(b) Show that $\ker\varphi$ is closed under multiplication and inverses:

Let $x, y \in \ker\varphi$. Then $\varphi(x) = \varphi(y) = 1_H$ and

$$\varphi(xy) = \varphi(x)\varphi(y) = 1_H \text{ yields } xy \in \ker\varphi$$

$$x^{-1} \in \ker\varphi \text{ since by (5b) } \varphi(x^{-1}) = \varphi(x)^{-1} = 1_H.$$

$\square$